

ExpEcon Methods: Intro to Hypothesis Testing and Fay & Proschan's (2010) "Perspectives"

ECON 8877

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First version thanks to Sungmin Park

Updated 2026-03-31

A Primer/Reminder on Hypothesis Tests

- Assumed model/DGP (“population”): $F(X; \theta)$, $\theta \in \Theta$
- Sample (r.v.): X . Often $X = (X_1, \dots, X_n)$. Realization: x
- Sample statistic: $W(X)$
 - Ex: 2 groups. $X_i = (Y_i^0, Y_i^1)$, $W(X) = \sum_i Y_i^0/n - \sum_i Y_i^1/n$
- Hypotheses: $H_0 : \theta \in \Theta_0$, $H_1 : \theta \in \Theta_1$
 - Here: Assume $\Theta_1 = \Theta \setminus \Theta_0$
- Do not reject H_0 if $W(x) \in W_0 \subseteq \mathbb{R}$
- Reject if $W(x) \in W_0^C$. Let $R = \{x : W(x) \in W_0^C\}$
 - Decision rule: $\delta(x; W_0^C) = 1$ if $W(x) \in W_0^C$
 - Decision rule: $\delta(x; W_0^C) = 0$ if $W(x) \notin W_0^C$
- Rejection region (in sample space): reject H_0 if $x \in R$
 - Pr(Type I Error at $\theta \in \Theta_0$) = $P_\theta(X \in R)$
 - Pr(Type II Error at $\theta \in \Theta_0^C$) = $P_\theta(X \notin R) = 1 - P_\theta(X \in R)$
- Power function: $\beta(\theta) = P_\theta(X \in R)$
 - Ideal: $\beta(\theta) = 0 \forall \theta \in \Theta_0$, $\beta(\theta) = 1 \forall \theta \in \Theta_0^C$

Primer: One-Sided Test

Example: $X = (X_1, \dots, X_n) \in \mathbb{R}^n$, iid $N(\theta, \sigma^2)$, σ^2 known.

- Question: $H_0 : \theta \leq \theta_0$ for some θ_0 (given by research question)
- Test statistic: $W(X) = \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}}$ (average “z-score”)
- Reject if $W(X) > c$ for some c (that you choose)
 - Why? Equivalent to LRT $\sup_{\theta \in \Theta_0} L(\theta|x) / \sup_{\theta} L(\theta|x) < \lambda$
 - Parametric distribution $\Rightarrow L(\theta|x)$ known
- How to pick c ? Want $\beta(\theta) \leq 0.05$ for all $\theta \leq \theta_0$. So

$$\begin{aligned}\beta(\theta) &= P_{\theta} \left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c \right) \\ &= P_{\theta} \left(\frac{\bar{X} - \theta}{\sigma/\sqrt{n}} > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right) \\ &= P_{\theta} \left(Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right) = 1 - \Phi \left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right)\end{aligned}$$

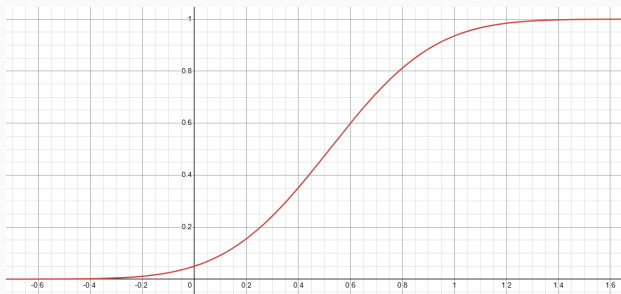
- Increasing in $\theta \Rightarrow$ Want $\beta(\theta_0) = 1 - \Phi(c) = 0.05 \Rightarrow$ Set $c = 1.645$

Primer: One-Sided Test

Example: $X = (X_1, \dots, X_n) \in \mathbb{R}^n$, iid $N(\theta, \sigma^2)$, σ^2 known.

Test: Reject if $W(X) = \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c = 1.645$

Graph: $\beta(\theta)$ for $\theta_0 = 0$, $n = 10$



Maximum type-I error (among $\theta \leq \theta_0 = 0$) is 0.05 (by design)

Size of test: $\alpha := \sup_{\theta \in \Theta_0} \beta(\theta)$

“I want 80% power.” OK but... at which θ ???

Primer: p -Values

So, what's a p -value?

- In general, just another statistic $p(X)$
- But it's an alternative (equivalent) way to run the same test
- But most commonly, rejection rule is $R = \{x : p(x) < \alpha\}$ where

$$p(x) = \sup_{\theta \in \Theta_0} P_{\theta}(W(X) \geq W(x))$$

- “Under H_0 , what the probability of a more-extreme $W(X)$?”
- Reject iff $p(x) < \alpha$
 - Decision rule: $\delta(x; \alpha) = 1$ if $p(x) < \alpha$, $\delta(x; \alpha) = 0$ if $p(x) \geq \alpha$
 - Previously you chose c , reject if $W(X) > c$
 - Now you choose α , reject if $p(X) < \alpha$
- This will generate a valid test for any α
- One-sided test: reject if $W(X) > w^{**}$
- Two-sided test: reject if $W(x) \notin (w^*, w^{**})$

The Example

Again, $X_i \sim N(\theta, \sigma^2)$, $H_0 : \theta \leq \theta_0 (= 0)$.

$$\begin{aligned} p(x) &= \sup_{\theta \in \Theta_0} P_{\theta}(W(X) \geq W(x)) \\ &= \sup_{\theta \in \Theta_0} P_{\theta} \left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \geq \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \right) \\ &= P_{\theta_0} \left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \geq \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \right) \\ &= P_{\theta_0} (Z \geq W(x)) \end{aligned}$$

- Usual rule: reject H_0 iff $p(x) < 0.05$
- Note: $p(x) < 0.05$ iff $W(x) > 1.645$
- So, reject if $W(x) > 1.645$. Same rule as before!!! $c = 1.645$
- Has same meaning regardless of $W(\cdot)$ and W_0
- p -value gives a useful measure of “how close you were” to rejecting/not rejecting

How To Simulate It

- In the normal example, $\beta(\theta)$ has analytic solution
- In general might not exist/too hard to solve
- We can simulate it! Steps:
 1. Set a grid of θ values
 2. Choose a sample size n and number of “runs” R
 3. At each “true” θ generate R iid samples of size n
 - r^{th} sample is $x_r^\theta = (x_{1,r}^\theta, \dots, x_{n,r}^\theta)$ where $x_{i,r}^\theta \stackrel{iid}{\sim} F(\cdot, \theta)$
 4. For each x_r^θ determine $\delta(x_r^\theta; \alpha)$
 5. Estimated power function: $\hat{\beta}(\theta) = \sum_{r=1}^R \frac{1}{R} \delta(x_r^\theta; \alpha)$
 6. Redo this for various $n, \alpha, W(x)$, whatever...
Plot them all and check:
 - 6.1 Is $\hat{\beta}(\theta) \leq 0.05$ when $\theta \in \Theta_0$?
 - 6.2 Which has the greatest $\hat{\beta}(\theta)$ when $\theta \notin \Theta_0$?
- Very useful for picking your actual sample size!

Example MATLAB Code

```
1 %% Set parameters
2 tnot = 0; % the cutoff theta_0
3 tgrid = [-2.5 -1 -0.5 -0.1 0 0.1 0.5 1 1.5 2 2.5 5 10]; %
   true means (theta)
4 n = 100;
5 c = 1.645;
6 sig = 10; %true std deviation
7 runs = 5000; %how many samples to generate for each t and n
```

Part 1: Setting Parameters

Example MATLAB Code

```
1 %% Now run the simulation
2 for ti=1:length(tgrid)
3     t = tgrid(ti); % current "true" theta
4     fdist = makedist('Normal','mu',t,'sigma',sig); %true
        dist'n
5     for r = 1:runs
6         xr = random(fdist,n,1); %rth sample, an nx1 vector
7         W(ti,r) = (mean(xr)-tnot)/(sig/sqrt(n)); %sample
            statistic
8         rej(ti,r) = W(ti,r) > c; %reject or not
9         pval(ti,r) = 1-normcdf(W(ti,r),tnot,1); %our p-value
10    end
11    rejavg(ti) = mean(rej(ti,:));
12    pvalavg(ti) = mean(pval(ti,:));
13 end
```

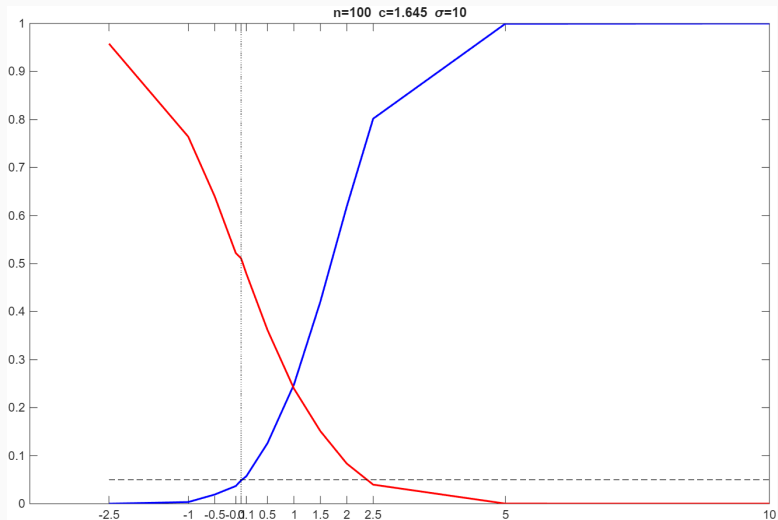
Part 2: Running the Simulation

Example MATLAB Code

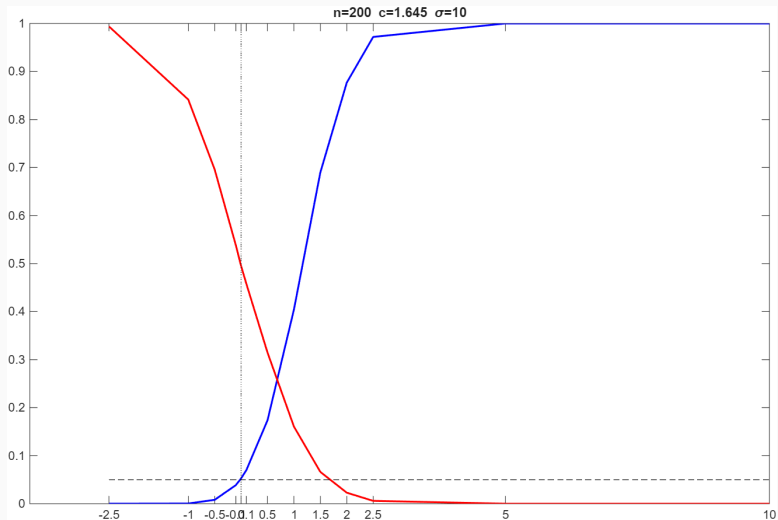
```
1 %% Now plot the output
2 figure; %open new figure
3 plot(tgrid,rejavg,"LineWidth",1.5,"Color","blue"); %plot
   rejavg (power)
4 title("n="+n+" c="+c+" \sigma="+sig);
5 xticks(tgrid);
6 ylim([0 1]);
7 hold on; %allows multiple plots
8 plot(tgrid,pvalavg,"LineWidth",1.5,"Color","red"); %plot
   pvalavg
9 plot(tgrid,0.05*ones(length(tgrid)),"LineStyle","--","Color
   ","k"); %plot y=0.05 as black dashed line
10 plot([tnot tnot],[0 1],"LineStyle",":","Color","k"); %plot x
   =0
11 hold off;
```

Part 3: Plotting Output

Example MATLAB Simulation



Example MATLAB Simulation



Primer: Two-Sided t -Test

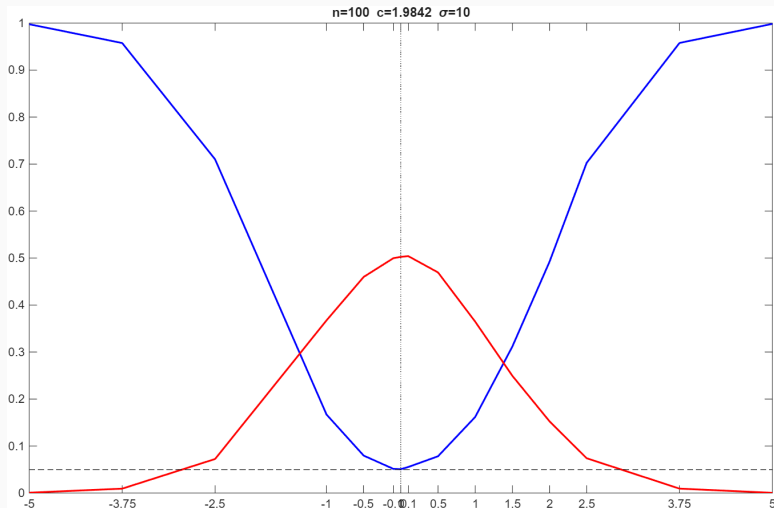
Example: $X = (X_1, \dots, X_n) \in \mathbb{R}^n$, iid $N(\theta, \sigma^2)$, σ^2 **NOT** known.

- Question: $H_0 : \theta = \theta_0, H_1 : \theta \neq \theta_0$
- Test statistic: $W(X) = \frac{\bar{X} - \theta_0}{\hat{\sigma}/\sqrt{n}}$ where $\hat{\sigma} = \sqrt{\sum_i \frac{1}{n-1} (X_i - \bar{X})^2}$
- Reject if $W(X) \notin [-c, c]$ for some c . How to pick c ?
- Want $\beta(\theta) \leq 0.05$ for all $\theta \in \Theta_0 = \{\theta_0\}$. So

$$\begin{aligned}\beta(\theta_0) &= 2 \cdot P_{\theta_0} \left(\frac{\bar{X} - \theta_0}{\hat{\sigma}/\sqrt{n}} > c \right) \\ &= 2 \cdot P_{\theta_0} \left(\frac{\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}}}{\sqrt{\frac{\sum_i (X_i - \bar{X})^2}{\sigma^2} \frac{1}{n-1}}} > c \right) \\ &= 2 \cdot P_{\theta_0} \left(\frac{Z}{\sqrt{\chi^2/(n-1)}} > c \right) = 2 \cdot (1 - T_{n-1}(c))\end{aligned}$$

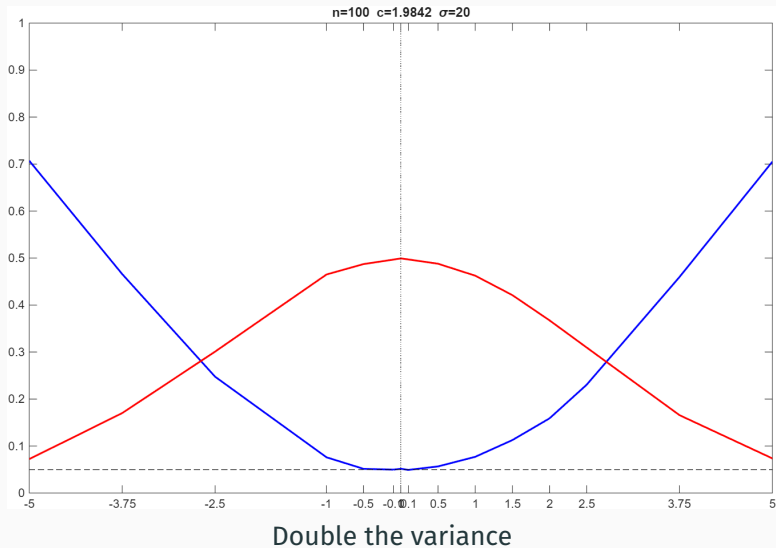
- Want $\beta(\theta_0) = 0.05 \Rightarrow$ Set $c = T_{n-1}^{-1}(0.975)$. Note: depends on n
- Does a closed-form solution exist for $\beta(\theta)$? Let's just simulate!!

Primer: Two-Sided Test



x-axis: True mean. Blue: Power function. Red: Average p-value.

Primer: Two-Sided Test



Comparing tests

- Suppose you have a class of tests \mathcal{C} for a fixed H_0
- Fix n . Test w/ power function $\beta \in \mathcal{C}$ is **uniformly most powerful (UMP)** if $\forall \beta' \in \mathcal{C}, \forall \theta \in \Theta_0^c, \beta(\theta) \geq \beta'(\theta)$
 - May not exist
- Test with β is asymptotically (uniformly) most powerful if it becomes UMP as $n \rightarrow \infty$
- A test is **valid** for size α if $\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha$
- It is **asymptotically valid** if it's valid as $n \rightarrow \infty$

Now We're Ready

OK, now onto Fay & Proschan (2010)...

Two Popular Statistical Tests

Two samples: $Y^0 = (Y_1^0, \dots, Y_n^0)$ and $Y^1 = (Y_1^1, \dots, Y_m^1)$

Assume $Y_i^0 \stackrel{iid}{\sim} F$ and $Y_j^1 \stackrel{iid}{\sim} G$. Is G "bigger" than F ?

In what sense? Mean? Median? FOSD? What does a given test measure?

- **Student's t -test:** reject if

$$\left| \frac{\hat{\mu}_1 - \hat{\mu}_0}{\hat{\sigma} \sqrt{\frac{1}{n} + \frac{1}{m}}} \right| > t_{n+m-2}^{-1}(1-\alpha/2)$$

where $\hat{\sigma}^2$ is the pooled sample variance and $t_d(\cdot)$ is the CDF of Student t distribution with degree of freedom d .

- Parametric. Test of means? Assumes normality?
- **Wilcoxon/Mann-Whitney rank-sum test:** reject if

$$\sum_{i=1}^n \sum_{j=1}^m S(Y_i^0, Y_j^1) < U_{n_0, n_1}^{-1}(1-\alpha/2) \quad \text{where } S(x, y) = \begin{cases} 1, & \text{if } x > y, \\ \frac{1}{2}, & \text{if } x = y, \\ 0, & \text{if } x < y. \end{cases}$$

- Non-parametric. Test of medians??? Assumes what???

Question

When is it appropriate to use **Wilcoxon-Mann-Whitney (WMW) test** or **t-test** to compare two samples?

- When is it **valid** & **consistent**? When is it **optimal**?

Answer

They are appropriate for different pairs of **null** and **alternative** hypotheses (“**perspectives**”)



Illustration

Illustration: 9th Grade Math Ability of Boys & Girls

Figure 1: Histograms of math ability

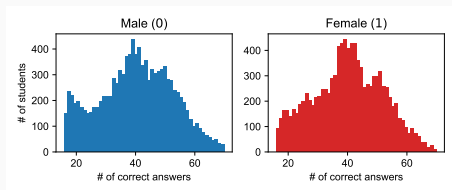


Table 1: Summary statistics of math ability

Statistic		Sample (j)	
		Male (0)	Female (1)
Obs.	n_j	10,887	10,557
Mean	$\hat{\mu}_j$	40.17	40.20
Median		40.44	40.36
Variance	$\hat{\sigma}_j^2$	152.00	134.74

Source: High School Longitudinal Study (HSLs) of 2009

- Assuming each obs is **independent**, should we use t -test? WMW test? To test what?
- Fay and Proschan (2010) say that the answer depends on your perspective(s).
- A **perspective** is a pair of null (H) and alternative (K) hypotheses.

One perspective you know from Stats 101

Perspective (Shift in normal distribution)

Let Y denote a random variable. The **shift-in-normal perspective**

is $H : \mathbb{E}_F(Y) = \mathbb{E}_G(Y)$ versus $K : \mathbb{E}_F(Y) \neq \mathbb{E}_G(Y)$, where F and G are two **normal** distributions with the **same variance**. (Difference must be in means.)

- **Student's t-test** (decision rule): Given data X and significance level α , reject H if

$$\left| \frac{\hat{\mu}_1 - \hat{\mu}_0}{\hat{\sigma} \sqrt{\frac{1}{n_1} + \frac{1}{n_0}}} \right| > t_{n-2}^{-1}(1-\alpha/2),$$

where $\hat{\sigma}^2$ is the pooled sample variance, $n = n_1 + n_0$, and $t_d(\cdot)$ is the CDF of Student t distribution with degree of freedom d .

- Under the above, Student's t-test is not only **valid** (α works as intended) but also **uniformly most powerful (UMP) unbiased**. It's also **asymptotically most powerful (AMP)**.

A relaxed perspective, also from Stats 101

Perspective (Behrens-Fisher)

The **Behrens-Fisher perspective** is

$$H : \mathbb{E}_F(Y) = \mathbb{E}_G(Y) \quad \text{versus} \quad K : \mathbb{E}_F(Y) \neq \mathbb{E}_G(Y),$$

where F and G are two **normal** distributions with possibly **different variances**.

- Under this relaxed perspective, Student's t -test is no longer valid because it pools the variances.
- **Welch's t -test** uses separate variance estimates, thus is **asymptotically valid** and **asymptotically most powerful**:

$$\left| \frac{\hat{\mu}_1 - \hat{\mu}_0}{\sqrt{\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_0^2}{n_0}}} \right| > t_{d_W}^{-1}(1-\alpha/2), \quad \text{where } d_W = \frac{\left(\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_0^2}{n_0}\right)^2}{\frac{(\hat{\sigma}_1^2/n_1)^2}{n_1-1} + \frac{(\hat{\sigma}_0^2/n_0)^2}{n_0-1}}$$

⇒ Each statistical test can have multiple valid perspectives. The authors call this idea the **Multiple perspective decision rules (MPDR) framework**

Even more relaxed perspective

Perspective (Distributions equal or not)

$$H : F = G \quad \text{versus} \quad K : F \neq G,$$

where F and G are any two distributions.

- Under this perspective, the t-tests are **asymptotically valid** and the WMW test is **valid**. But neither are **consistent!** (power approaches 1 as $n \rightarrow \infty$)

- The WMW test (or Mann-Whitney U test or Wilcoxon rank-sum test) is to reject if

$$\sum_{i=1}^n \sum_{j=1}^m S(Y_i^0, Y_j^1) < U_{n_0, n_1}^{-1}(1-\alpha/2) \quad \text{where } S(x, y) = \begin{cases} 1, & \text{if } x > y, \\ \frac{1}{2}, & \text{if } x = y, \\ 0, & \text{if } x < y. \end{cases}$$

- Neither t-tests nor WMW test reject the null hypothesis for the 9th-graders' data

Philosophy behind the MPDR framework

- The **Multiple perspective decision rules (MPDR) framework** has practical value because it suits the nature of scientific theories.
- A **scientific theory** is often a **vague idea** or a **qualitative result** that can be described by more than one statistical model.
 - In biological sciences, for example, the Physicians' Health Study (PHS) aims to test a theory that says **prolonged low-dose aspirin** decreases **cardiovascular mortality**.
 - Researchers testing this theory assume a particular statistical model to formulate the null hypothesis, but that model is **just one way** of representing the data's randomness.
- So we should consider the **set of possible statistical assumptions** behind a scientific theory to assess which statistical tests (decision rules) are the most useful.

Framework

Terminology

Data	$X \in \mathcal{X}$, where \mathcal{X} is the sample space. Write X_n to denote number of observations n	
“Probability model”	A distribution $P \in \mathcal{P}$ on \mathcal{X} , where $\mathcal{P} = \{P_\theta \theta \in \Theta\}$ with a given parameter space Θ	
Null hypothesis	$H = \{P_\theta \theta \in \Theta_H\}$	
Alternative hypothesis	$K = \{P_\theta \theta \in \Theta_K\}$	(Θ_H and Θ_K are disjoint subsets of Θ)
“Assumption”	$A = (\mathcal{X}, H, K)$	
Decision rule (test)	$\delta(X, \alpha) \in \{0 \text{ (not reject), } 1 \text{ (reject)}\}$, for all data $X \in \mathcal{X}$ and significance level $\alpha \in (0, 0.5)$	

Terminology about decision rule (test) δ

“Power” $Pow[\delta(X_n, \alpha); \theta] = \Pr[\delta(X_n, \alpha) = 1; \theta]$ (Probability of rejecting)

“Size” $\alpha_n^* = \sup_{\theta \in \Theta_H} Pow[\delta(X_n, \alpha); \theta]$. (Max. prob. of rejecting given null)

Validity A test δ is **valid** if $\alpha_n^* \leq \alpha$ for all n .

A test δ is **uniformly asymptotically valid (UAV)** if $\limsup_{n \rightarrow \infty} \alpha_n^* \leq \alpha$.

A test δ is **pointwise asymptotically valid (PAV)** if, for all $\theta \in \Theta_H$,

$$\limsup_{n \rightarrow \infty} Pow[\delta(X_n, \alpha); \theta] \leq \alpha.$$

p-value $p(X) = \inf\{\alpha' : \delta(X, \alpha') = 1\}$ (the strictest α' that rejects)

Terminology about optimal decision rules

Bias A test δ is **unbiased** if, for all $\theta \in \Theta_K$, power \geq size.

Consistency A test δ is **consistent** if, for all $\theta \in \Theta_K$, the power approaches 1 as $n \rightarrow \infty$.

Optimality A test δ is **uniformly most powerful (UMP)** if, $\forall \delta'$ and $\forall \theta \in \Theta_K$,

$$\text{Pow}[\delta(X, \alpha); \theta] \geq \text{Pow}[\delta'(X, \alpha); \theta].$$

A test is **UMP unbiased** if it is UMP among all unbiased tests.

A test is **asymptotically most powerful (AMP)** if, as θ_n approaches θ_0 ,

$$\limsup_{n \rightarrow \infty} \text{Pow}[\delta(X_n, \alpha); \theta_n] - \text{Pow}[\delta'(X_n, \alpha); \theta_n] \geq 0$$
as $\theta_n \in \Theta_K$ approaches $\theta_0 \in \Theta_H$.

Perspectives

Perspective (Difference in means; same null distribution)

$$H = \{F, G : F = G\}$$

$$K = \{F, G : \mathbb{E}_F(Y) \neq \mathbb{E}_G(Y)\}$$

- Weird (“**focusing**”) perspective because it leaves out many pairs of distributions
- Still, the alternative hypotheses K is a pretty large set
- The WMW test is **valid but inconsistent**
- The paper doesn't mention how the t-tests fare, but they are likely inconsistent, too.
- So, don't take this perspective.

Perspective (Stochastic ordering)

Let Ψ_C denote the set of continuous distributions. Write $F <_{st} G$ if G has **first-order stochastic dominance** over F (i.e. $F(y) \geq G(y)$ for all y and $F(y) > G(y)$ for some y).

$$H = \{F, G : F = G; F \in \Psi_C\}$$

$$K = \{F, G : F <_{st} G \text{ or } G <_{st} F; F, G \in \Psi_C\}$$

- Under this perspective, the WMW test is **valid** and **consistent** (Mann and Whitney, 1947). It's also **unbiased** (Lehmann, 1951)
- The t-tests (both Student's and Welch's) are **asymptotically valid** and **consistent**
- So, both the WMW test and t-tests work under this perspective!

Perspective (Mann-Whitney Functional)

Let $Y_F \sim F$ and $Y_G \sim G$. Define the *Mann-Whitney functional* ϕ as

$$\phi(F, G) = \Pr[Y_F > Y_G] + \frac{1}{2} \Pr[Y_F = Y_G]$$

The *Mann-Whitney functional perspective* is

$$H = \{F, G : F = G; F \in \Psi_C\},$$

$$K = \{F, G : \phi(F, G) \neq \frac{1}{2}; F, G \in \Psi_C\}.$$

- A natural perspective by construction. Especially appropriate for ordinal data
- The WMW test is valid and consistent, whereas the t-tests are inconsistent
- So don't use t-tests under this perspective. Use the WMW test

Perspective (Distribution equal or not)

$$H = \{F, G : F = G\}$$

$$K = \{F, G : F \neq G\}$$

- The WMW test is valid but inconsistent. The t-tests are asymptotically valid but inconsistent.
- If you take this perspective, find a different test like Kolmogorov-Smirnov

Perspectives 5–8: Shifts & scale in distributions

Let Ψ_L , Ψ_C , and Ψ_{LG} denote the sets of **logistic**, **continuous**, and **log-gamma** distributions.

Let Ψ_{D_k} denote the set of discrete distributions with sample space $\{1, 2, \dots, k\}$

Perspective (Shift in logistic distribution)

$$H = \{F, G : F = G; F \in \Psi_L\}$$

$$K = \{F, G : G(y) = F(y + \Delta); \Delta \neq 0; F \in \Psi_L\}$$

Perspective (Shift in continuous distribution)

$$H = \{F, G : F = G; F \in \Psi_C\}$$

$$K = \{F, G : G(y) = F(y + \Delta); \Delta \neq 0; F \in \Psi_C\}$$

Perspective (Shift in log-gamma distribution)

$$H = \{F, G : F = G; F \in \Psi_{LG}\}$$

$$K = \{F, G : G(y) = F(y + \Delta); \Delta \neq 0; F \in \Psi_{LG}\}$$

Perspective (Proportional odds)

$$H = \{F, G : F = G; F \in \Psi_{D_k}\}$$

$$K = \{F, G : \frac{F(y)}{1-F(y)} = \frac{G(y)}{1-G(y)} \Delta; \Delta \neq 1; F \in \Psi_{D_k}\}$$

- WMW: valid and consistent. t-tests: asymp. valid and consistent.

Perspective 11: Differences in means assuming normality with same variance

Perspective (Shift in normal distribution)

$$H = \{F, G : F = G; F \in \Psi_N\}$$

$$K = \{F, G : G(y) = F(y + \Delta); \Delta \neq 0; F \in \Psi_N\}$$

where Ψ_N is the set of normal distributions.

- The first perspective you've seen at the beginning.
- The WMW test and the Student's t-test are **valid and consistent**. The Student's t-test is **optimal**, because it is **UMP unbiased** and **asymptotically most powerful**. The Welch's t-test is **asymptotically valid and consistent**.

Perspective 14: Differences in means assuming normality with different variance

Perspective (Behrens-Fisher: Difference in normal means, different variances)

$$H = \{F, G : \mathbb{E}_F(Y) = \mathbb{E}_G(Y); F, G \in \Psi_N\}$$

$$K = \{F, G : \mathbb{E}_F(Y) \neq \mathbb{E}_G(Y); F, G \in \Psi_N\}$$

where Ψ_N is the set of normal distributions.

- Both the WMW test and the Student's t-test are **invalid and inconsistent**
- Welch's t-test is **uniformly asymptotically valid** and **consistent**
- So, use Welch's t-test if you take this perspective... but better ones exist:

Perspectives 12–13: Differences in means without assuming normality

Perspective (Finite variances)

$$H = \{F, G : F = G; F \in \Psi_{fV}\}$$

$$K = \{F, G : \mathbb{E}_F(Y) \neq \mathbb{E}_G(Y); F, G \in \Psi_{fV}\}$$

where Ψ_{fV} is the set of distributions with finite variances.

- The WMW test is **valid but inconsistent**
- The t-tests are **pointwise asymptotically valid** and **consistent**

Perspective (Finite 4th moments)

$$H = \{F, G : F = G; F \in \Psi_{B\epsilon}\}$$

$$K = \{F, G : \mathbb{E}_F(Y) \neq \mathbb{E}_G(Y); F, G \in \Psi_{B\epsilon}\}$$

where $\Psi_{B\epsilon}$ is the set of distributions with $\text{Var}(Y) \geq \epsilon > 0$ and $\mathbb{E}(Y^4) \leq B < \infty$.

- The WMW test is **valid but inconsistent**
- The t-tests are **uniformly asymptotically valid** and **consistent**

⇒ t-tests are clearly preferable in large samples

Perspective 15: Seemingly natural but invalid perspective

Perspective (Difference in means; any distributions)

$$H = \{F, G : \mathbb{E}_F(Y) = \mathbb{E}_G(Y)\}$$

$$K = \{F, G : \mathbb{E}_F(Y) \neq \mathbb{E}_G(Y)\}$$

- There **exists no valid decision rule** with some power greater than its significant level
- If you take this loose perspective, nothing works!
- Your perspective needs more structure

Discussion

Takeaways

So... WMW test or t-test?

- It's important to identify your perspective first! Be **precise!**
- t-test is usually only asymptotically valid...
- In the math ability example, maybe use **Welch's t-test** since $n, m \geq 10,000$
- But depending on the application, the **WMW test** may be more appropriate
 - For example, if the variable is **ordinal**. Also, the authors argue that the WMW test is often more powerful than the t-tests in **small samples**
- In any case, the decision should not depend on whether the data look normally distributed or not, because there are valid perspectives without the normality assumption
- But, stay tuned for the permutation test!

The End!



Suggested Reading

Michael P. Fay and Michael A. Proschan. Wilcoxon-Mann-Whitney or t-test? On assumptions for hypothesis tests and multiple interpretations of decision rules. *Statistics Surveys*, 4:1–39, 2010.