

Chapter 6: Mixed Strategies

ECON 5001

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Based on: *Game Theory* by Giacomo Bonanno

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Where We Left Off

Chapter 2: some games have **no** Nash equilibrium

- Matching Pennies, penalty kicks, play-calling...
- Being predictable gets you exploited

Chapter 5: **expected utility** ranks lotteries

- $u(o)$: outcome utility $U(L)$: expected utility

$$U(L) = \sum_i p_i u(o_i)$$

Our goal:

1. Let players **randomize**
2. Use expected utility to evaluate strategies in the face of a random opponent
3. Redefine equilibria to include random play
 - Show that, with this definition, it always exists

MobLab: “Matrix: Instructor Specified” ×2

- MobLab time!
- You’ll be paired with someone. Row player or Column player
- Two rounds, new partner each round. Each round played once

		Round 1	
		Column	
Row	Top	80, 40	40, 80
	Bottom	40, 80	80, 40

		Round 2	
		Column	
Row	Top	320, 40	40, 80
	Bottom	40, 80	80, 40

Feel free to talk! And play it the way you’d actually play it.

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Round 1

		Column	
		Left	Right
Row	Top	80, 40	40, 80
	Bottom	40, 80	80, 40

Round 2

		Column	
		Left	Right
Row	Top	320, 40	40, 80
	Bottom	40, 80	80, 40

Who should respond to 320? Row? Column? Both?

MobLab: “Matrix: Instructor Specified” ×2

- MobLab time!
- You’ll be paired with someone. Row player or Column player
- Two rounds, new partner each round. Each round played once

Round 1

		Column	
		Left	Right
Row	Top	80, 40	40, 80
	Bottom	40, 80	80, 40

Round 2

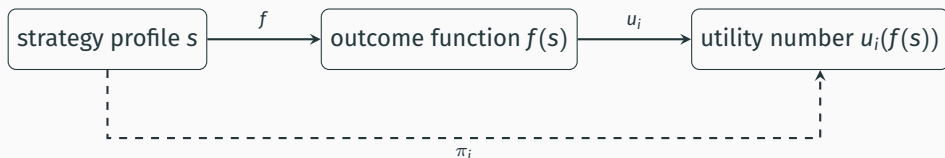
		Column	
		Left	Right
Row	Top	320, 40	40, 80
	Bottom	40, 80	80, 40

Who should respond to 320? Row? Column? Both?
Paradoxical prediction: **Column** responds, **not** row

Strategic-Form Games With Randomness

Strategic-Form Games

Same as Chapter 2:



- Payoff $\pi_i(s)$ gives i 's utility for the outcome generated by s
 - No need to specify outcomes, just payoffs
- Utility numbers are **expected utility values**, like little $u(o_i)$
 - Remember: only linear rescaling is allowed
- Bonanno: “games with cardinal payoffs”

How To Evaluate Randomness?

	Rock	Paper	Scissors
Rock	<i>Draw</i>	<i>Lose</i>	<i>Win</i>
Paper	<i>Win</i>	<i>Draw</i>	<i>Lose</i>
Scissors	<i>Lose</i>	<i>Win</i>	<i>Draw</i>

P1's outcome shown

How To Evaluate Randomness?

	Rock	Paper	Scissors
Rock	Draw	Lose	Win
Paper	Win	Draw	Lose
Scissors	Lose	Win	Draw

P1's outcome shown

- Player 1: Win $\succ$ Draw $\succ$ Lose

How To Evaluate Randomness?

	Rock	Paper	Scissors
Rock	<i>Draw</i>	<i>Lose</i>	<i>Win</i>
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Scissors	<i>Lose</i>	<i>Win</i>	<i>Draw</i>

P1's outcome shown

- Player 1: $\text{Win} \succ \text{Draw} \succ \text{Lose}$
- Let's try: $u(\text{Win}) = 1, u(\text{Draw}) = 0, u(\text{Lose}) = -1$

How To Evaluate Randomness?

	Rock	Paper	Scissors
Rock	<i>Draw</i>	<i>Lose</i>	<i>Win</i>
Paper	<i>Win</i>	<i>Draw</i>	<i>Lose</i>
Scissors	<i>Lose</i>	<i>Win</i>	<i>Draw</i>

P1's outcome shown

- Player 1: $\text{Win} \succ \text{Draw} \succ \text{Lose}$
- Let's try: $u(\text{Win}) = 1$, $u(\text{Draw}) = 0$, $u(\text{Lose}) = -1$
- And suppose your opponent is playing each strategy with $1/3$ chance

How To Evaluate Randomness?

	R (1/3)	P (1/3)	S (1/3)
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

P1's outcome utility ($u(o)$) shown

Use expected utility to evaluate strategies:

- $U(\text{Rock}) = (1/3) \cdot 0 + (1/3) \cdot (-1) + (1/3) \cdot 1 = 0$
- $U(\text{Paper}) = (1/3) \cdot 1 + (1/3) \cdot 0 + (1/3) \cdot (-1) = 0$
- $U(\text{Scissors}) = (1/3) \cdot (-1) + (1/3) \cdot 1 + (1/3) \cdot 0 = 0$

How To Evaluate Randomness?

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- $U(\text{Scissors}) = (1/3) \cdot (-1) + (1/3) \cdot 1 + (1/3) \cdot 0 = 0$

You're exactly indifferent! All strategies are best replies.

How To Evaluate Randomness?

	R (1/3)	P (1/3)	S (1/3)
Rock	0	-1	1
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- $U(\text{Scissors}) = (1/3) \cdot (-1) + (1/3) \cdot 1 + (1/3) \cdot 0 = 0$

You're exactly indifferent! All strategies are best replies.

Thus, you're happy to randomize between them. ← **Indifference Principle**

Football Example: Run vs. Pass

Another source of randomness: **Random outcomes**

Example: Going for it on 4th down:

- Ohio State offense calls: **Run** or **Pass**
- Michigan's defense calls: **Stop Run** or **Stop Pass**
- Simultaneous: neither sees the other's call

Even after both calls, the play can work or fail

Outcome: a **lottery** over {Score, Fail}

Sometimes the Outcome Is a Lottery

	Stop Run	Stop Pass
Run	$\begin{pmatrix} \text{Score} & \text{Fail} \\ 0.4 & 0.6 \end{pmatrix}$	$\begin{pmatrix} \text{Score} & \text{Fail} \\ 0.8 & 0.2 \end{pmatrix}$
Pass	$\begin{pmatrix} \text{Score} & \text{Fail} \\ 0.9 & 0.1 \end{pmatrix}$	$\begin{pmatrix} \text{Score} & \text{Fail} \\ 0.3 & 0.7 \end{pmatrix}$

What is the payoff in each cell?

Sometimes the Outcome Is a Lottery

	Stop Run	Stop Pass
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What is the payoff in each cell? Use **expected utility**

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What is the payoff in each cell? Use **expected utility**

Normalize OSU's payoff: $u(\text{Score}) = 1$, $u(\text{Fail}) = 0$

$$U \begin{pmatrix} \text{Score} & \text{Fail} \\ 0.4 & 0.6 \end{pmatrix} = \underline{0.4(1) + 0.6(0) = 0.4} = \text{Pr}(\text{Score})$$

The Reduced Game

Multiply by 100 (linear rescaling). Payoff = % chance of each team's goal

		Michigan	
		Stop Run	Stop Pass
Ohio State	Run	40, 60	80, 20
	Pass	90, 10	30, 70

Rule: If outcomes are random, use expected utility to calculate payoff

Convention: Skip the outcomes/lotteries, just write the payoff number in each cell

Bonanno §6.1 describes this in more detail. Skip it for this class.

Mixed Strategies

Any Nash Equilibrium?

		Michigan	
		Stop Run	Stop Pass
Ohio State	Run	40, 60	80, 20
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No cell has both underlined. No (pure) Nash equilibrium!

Any Nash Equilibrium?

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What's optimal here?

Be unpredictable: **Randomize**

Any Nash Equilibrium?

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No cell has both underlined. No (pure) Nash equilibrium!

What's optimal here?

Be unpredictable: **Randomize**

But with what probabilities?? 50-50??

Best Reply to a Mixed Strategy

Suppose Michigan plays $\begin{pmatrix} \text{Stop Run} & \text{Stop Pass} \\ 0.8 & 0.2 \end{pmatrix}$. What should OSU do?

		Michigan	
		Stop Run	Stop Pass
Ohio State	Run	40, 60	80, 20
	Pass	90, 10	30, 70

$$\Pi_{OSU}(\text{Run}, q = 0.8) =$$

$$\Pi_{OSU}(\text{Pass}, q = 0.8) =$$

If $q = 0.8$: Run payoff: _____, Pass payoff _____

OSU's best reply to $q = 0.8$: _____

Best Reply to a Mixed Strategy

Suppose Michigan plays $\begin{pmatrix} \text{Stop Run} & \text{Stop Pass} \\ q & 1 - q \end{pmatrix}$. What should OSU do?

		Michigan	
		Stop Run	Stop Pass
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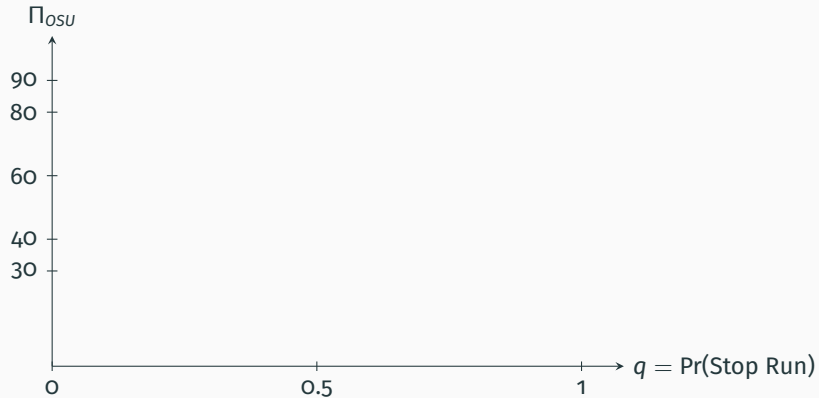
$$\Pi_{OSU}(\text{Run}, q) =$$

$$\Pi_{OSU}(\text{Pass}, q) =$$

If $q = 0.5$: Run gives _____, Pass gives _____

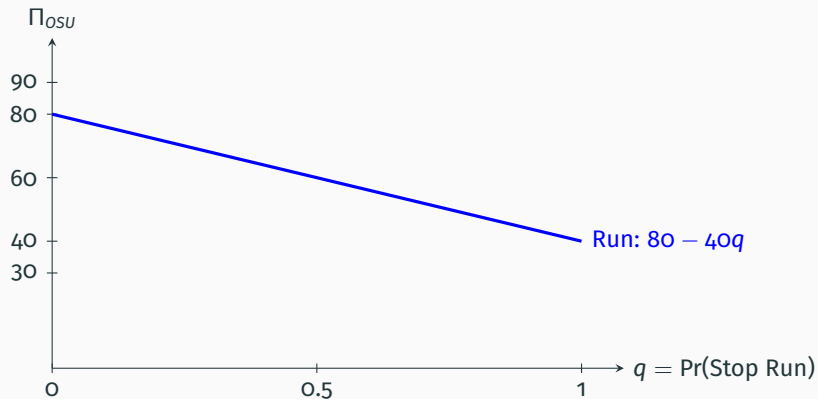
OSU's best reply: _____

Seeing It in a Graph



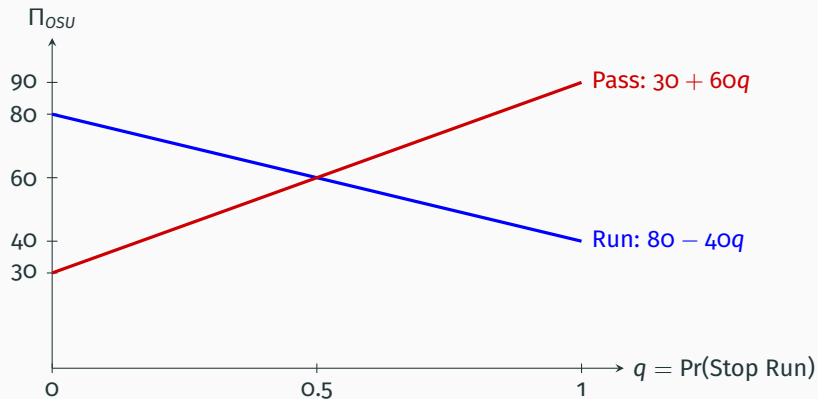
Plot each pure strategy's expected payoff against q

Seeing It in a Graph



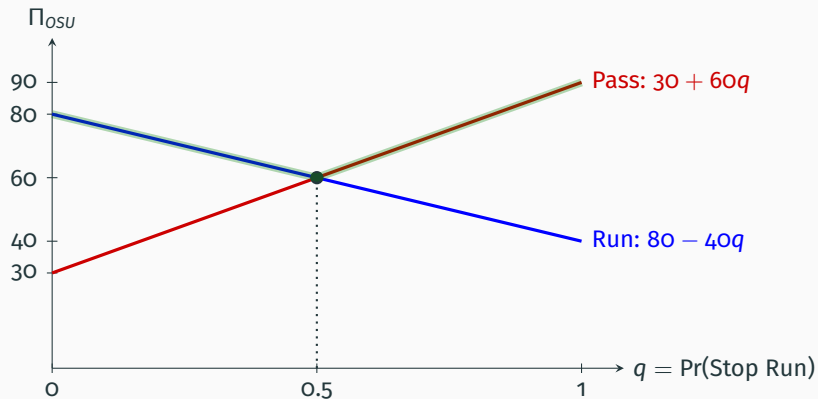
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Seeing It in a Graph



Plot each pure strategy's expected payoff against q

Seeing It in a Graph



Best reply = the higher line. They cross at $q = \underline{\hspace{2cm}}$

$$BR_{OSU}(q) = \begin{cases} \text{Run} & \text{if } q < \frac{1}{2} \\ \text{---} & \text{if } q = \frac{1}{2} \\ \text{Pass} & \text{if } q > \frac{1}{2} \end{cases}$$

Fact: A mix of Run and Pass is a best reply **only** when Run and Pass tie

- Otherwise, shift weight to the better one!

Fact: When they tie, **every** mix is a best reply

In general: σ_i is a best reply $\iff$ every pure strategy it uses is a best reply

Definition: Mixed Strategy

A **mixed strategy** σ_i is a probability over each of player i 's strategies

Ex: $\sigma_1 = \begin{pmatrix} \text{Run} & \text{Pass} \\ p & 1-p \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} \text{Stop Run} & \text{Stop Pass} \\ q & 1-q \end{pmatrix}$

Notation:

- $\sigma_i(s_i)$: probability that σ_i puts on s_i . Rule: $\sum_{s_i} \sigma_i(s_i) = 1$
 - $\sigma_1(\text{Run}) = p, \sigma_1(\text{Pass}) = 1 - p$
 - $\sigma_2(\text{Stop Run}) = q, \sigma_2(\text{Stop Pass}) = 1 - q$
- Σ_i : set of all mixed strategies of player i . So $\sigma_i \in \Sigma_i$
- A non-mixed strategy $s_i \in S_i$ is called a **pure strategy**
 - Pure strategy = mixed strategy with probability 1 on one s_i

Assume: players randomize **independently** (no correlation)

Expected Payoff of a Mixed Profile

A mixed strategy profile $\sigma = (\sigma_1, \sigma_2)$ generates a lottery over the **cells**

Probability of cell $s = (s_1, s_2)$: $\sigma(s) = \sigma_1(s_1) \times \sigma_2(s_2)$ (independence)

Expected payoff: weight each cell's payoff by the chance of landing in it

$$\underbrace{\Pi_i(\sigma)}_{\text{expected payoff}} = \sum_{s \in S} \underbrace{\sigma(s)}_{\text{prob. of cell } s} \cdot \underbrace{\pi_i(s)}_{\text{payoff in cell } s}$$

Capital Π : payoff from a mixed profile. Small π : from a pure profile

Same idea as U vs. u in Chapter 5

Definition: Mixed-Strategy Nash Equilibrium

In words: every player's mix is a best reply to the other players' mixes

A mixed profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a **Nash equilibrium** if, for every player i ,

$$\underbrace{\Pi_i(\sigma_i^*, \sigma_{-i}^*)}_{\text{payoff from equilibrium mix}} \geq \underbrace{\Pi_i(\sigma_i, \sigma_{-i}^*)}_{\text{payoff from any other mix}} \quad \text{for all } \sigma_i \in \Sigma_i$$

Equivalently: $\sigma_i^* \in BR_i(\sigma_{-i}^*)$ for every i

A pure-strategy NE is still a NE when mixing is allowed (Bonanno p. 211).

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Same as Chapter 2: $s_i^* \in BR_i(s_{-i}^*)$ for every i

A pure-strategy NE is still a NE when mixing is allowed (Bonanno p. 211).

Nash's Theorem

Theorem (Nash's 1951 PhD Dissertation at Princeton)

Every finite game has **at least one** Nash equilibrium in mixed strategies.

Finite game: finitely many players, each with finitely many strategies.

- Matching Pennies: no pure NE, but a mixed one
- Run vs. Pass: no pure NE, but a mixed one

Proof uses “Kakutani’s fixed-point theorem”. (Not covered here.)

Recall: We assume players maximize expected utility.

Finding Mixed Equilibria

The Indifference Principle

Indifference Principle: If you mix between two strategies in equilibrium, they must have the same expected payoff. (If not, you'd switch to playing the better one for sure!)

If σ_i^* puts positive probability on both s_i and s'_i , then

$$\underbrace{\Pi_i(s_i, \sigma_{-i}^*)}_{\text{payoff from } s_i} = \underbrace{\Pi_i(s'_i, \sigma_{-i}^*)}_{\text{payoff from } s'_i} = \underbrace{\Pi_i(\sigma^*)}_{\text{equilibrium payoff}}$$

Implication:

1. If I'm mixing between two strategies in equilibrium, it's because I'm indifferent
2. Thus, you must be mixing in a way that **makes me indifferent**
3. And if you're mixing, I'm also **making you indifferent**

So, mixed strategy equilibrium $\Rightarrow$ designed to make both players **indifferent**

An Example

		P2		
		L	C	R
P1	T	1, 0	0, 1	2, -1
	M	0, 2	1, 0	2, -1
	B	-1, 0	-1, 0	1, 3

B is strictly dominated by *M*

An Example

		P2		
		L	C	R
P1	T	1, 0	0, 1	2, -1
	M	0, 2	1, 0	2, -1
	B	-1, 0	-1, 0	1, 3

Then R is strictly dominated by C .
So, reduce the game to $\{T, M\} \times \{L, C\}$

An Example

		P2		
		L	C	R
P1	T	1, 0	0, 1	2, -1
	M	0, 2	1, 0	2, -1
	B	-1, 0	-1, 0	1, 3

Suppose $Pr(L) = q$ and $Pr(C) = 1 - q$. (We know $Pr(R) = 0$)

When is P1 indifferent between T and M?

$$\Pi_1(T) = q(1) + (1 - q)(0) = q$$

$$\Pi_1(M) = q(0) + (1 - q)1 = 1 - q$$

$$\Pi_1(T) = \Pi_1(M) \Rightarrow q = 1 - q \Rightarrow q = 1/2$$

Prediction: P1 plays $q = 1/2$

An Example

		P2		
		L	C	R
P1	T	1, 0	0, 1	2, -1
	M	0, 2	1, 0	2, -1
	B	-1, 0	-1, 0	1, 3

Suppose $Pr(T) = p$ and $Pr(M) = 1 - p$. (We know $Pr(B) = 0$)

When is P2 indifferent between L and C?

$$\Pi_2(L) = p(0) + (1 - p)(2) = 2 - 2p$$

$$\Pi_2(C) = p(1) + (1 - p)0 = p$$

$$\Pi_2(T) = \Pi_2(M) \Rightarrow p = 2 - 2p \Rightarrow 3p = 2 \Rightarrow p = 2/3$$

Prediction: P2 plays $p = 2/3$

Recipe: Mixed NE in a 2×2

1. Run IDSDS first. Dominated strategies get probability 0
2. Underline. Note any pure NE
3. P2 plays $q, 1 - q$. Find the q that makes **P1** indifferent
 - **Notice:** P2 is making P1 indifferent!
4. P1 plays $p, 1 - p$. Find the p that makes **P2** indifferent
 - **Notice:** P1 is making P2 indifferent!
5. Both strictly between 0 and 1? That's a mixed NE

Solving the Two-Point Game

q Michigan $1-q$
 Stop Run Stop Pass

Ohio State	Run	40, <u>60</u>	<u>80</u> , 20
Pass	Pass	<u>90</u> , 10	30, <u>70</u>

p
 $1-p$

Step 1: Michigan's q makes OSU indifferent

$$\text{Run: } q \cdot 40 + (1-q) \cdot 80 = 40q + 80 - 80q = 80 - 40q$$

$$\text{Pass: } q \cdot 90 + (1-q) \cdot 30 = 90q + 30 - 30q = 30 + 60q$$

$$q^* = \frac{1}{2}$$

$$80 - 40q = 30 + 60q$$

$$50 = 100q$$

$$q = \frac{1}{2}$$

Solving the Two-Point Game

Handwritten notes: $q = 1/2$, $1-q = 1/2$

		Michigan	
		Stop Run	Stop Pass
Ohio State	Run	0.3 $\underline{40}, 60$	0.3 $\underline{80}, 20$
	Pass	0.2 $90, 10$	0.2 $30, 70$

Handwritten notes on the table: 0.6 (circled) for Run, 0.4 for Pass, 0.5 for Stop Run, 0.5 for Stop Pass.

Step 2: OSU's p makes Michigan indifferent

Stop Run: $p(60) + (1-p)10 = 60p + 10 - 10p = 50p + 10$

Stop Pass: $p(20) + (1-p)70 = 20p + 70 - 70p = -50p + 70$

$p^* = 0.6$

Mixed NE:

$\sigma^* = ((0.6, 0.4), (\frac{1}{2}, \frac{1}{2}))$

OSU scores 60 % of the time

$50p + 10 = -50p + 70$

$Pr(\text{Score}) = 0.3(40) + 0.3(80) + 0.2(90) + 0.2(30)$
 $12 + 24 + 18 + 6 = 60$

$100p = 60$
 $p = 0.6$

The Unsettling Part of MSNE

Consider Rock Paper Scissors (RPS) whose MSNE is $\sigma^* = ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$

- In the MSNE you're indifferent between Rock, Paper, and Scissors.
- But then why do you play $(1/3, 1/3, 1/3)$?? You have **no incentive** either way!

Nash reasoning:

- When considering deviating, you assume **they don't change** their strategy σ_{-i}^* .
- If they're fixed at $(1/3, 1/3, 1/3)$ then you don't care what you play.
- Deviating away from $(1/3, 1/3, 1/3)$ doesn't help or hurt you.

“Dynamic” reasoning:

- If you deviate away from $(1/3, 1/3, 1/3)$ then they'll respond by moving away from $(1/3, 1/3, 1/3)$ to take advantage of you
- ...and then you'd change your strategy, and then so would they...
- Only “stable” point: Mixed Strategy Nash equilibrium σ^*
 - Caveat: that dynamic process won't *converge* to σ^* ... it'll cycle forever
 - Stable equilibrium vs. unstable equilibrium.

Back to MobLab

Round 1

$\frac{1}{2}$ $\frac{1}{2}$ Column 1-2
Left Right

Top	80, 40	40, 80
Bottom	40, 80	80, 40

Row

Top
Bottom

Round 1 mixed NE:

$$\begin{aligned}
 T: 80q + (1-q)40 &= 80q + 40 - 40q = 40q + 40 \\
 B: 40q + (1-q)80 &= 40q + 80 - 80q = 80 - 40q
 \end{aligned}$$

Round 2 mixed NE:

$$80q = 40 \implies q = \frac{1}{2}$$

Only Row's payoff changed. Whose mix changes in the NE??

Round 2

Column

Left Right

Top	320, 40	40, 80
Bottom	40, 80	80, 40

Row

Top
Bottom

$$L: 40p + (1-p)80 = 80 - 40p$$

$$R: 80p + (1-p)40 = 40p + 40$$

$$80 - 40p = 40p + 40$$

$$40 = 80p$$

$$p = \frac{1}{2}$$

Back to MobLab

$\frac{1}{2}$ $\frac{1}{2}$ Row

Round 1

	$\frac{1}{2}$ Column $\frac{1}{2}$	
	Left	Right
Top	80, 40	40, 80
Bottom	40, 80	80, 40

$\frac{1}{2} = p$ $\frac{1}{2} = 1-p$ Row

Round 2

	$\frac{1}{2}$ Column $\frac{7}{8}$	
	Left	Right
Top	320, 40	40, 80
Bottom	40, 80	80, 40

Round 1 mixed NE:

$$T: 320p + (1-p)40 = 40 + 280p$$

$$B: 40p + (1-p)80 = 80 - 40p$$

Round 2 mixed NE:

$$40 + 280p = 80 - 40p$$

$$320p = 40$$

$$p = \frac{40}{320} = \frac{1}{8} = \frac{2}{16} = \frac{1}{8} = 0.125$$

Only Row's payoff changed. Whose mix changes in the NE??

Col

$$L: 40p + 80(1-p) = 80 - 40p$$

$$R: 80p + 40(1-p) = 40p + 40$$

$$80 - 40p = 40p + 40$$

$$40 = 80p$$

$$p = \frac{1}{2}$$

	% Rows playing Top		% Columns playing Left	
	Round 1	Round 2	Round 1	Round 2
Nash prediction	50	50	50	12.5
Goeree & Holt (2001)	≈ 50	96	≈ 50	16

Lab result: Rows chase the 320. NE says Row's own payoffs shouldn't matter!
Columns anticipate it and play Left even less

Not in the book. Source: Goeree & Holt (2001), "Ten Little Treasures of Game Theory and Ten Intuitive Contradictions," *American Economic Review* 91(5).

Pure and Mixed Together

The Sunday Morning Game Revisited

Alex and Blair, dead phones: Buckeye Donuts or HangOverEasy?

		Blair	
		BuckyDs	HangOE
Alex	BuckyDs	4, 3	2, 1
	HangOE	1, 2	3, 4

Now read the payoffs as **cardinal** (outcome utilities)

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Alex	BuckyDs	<u>4</u> , <u>3</u>	2, 1
	HangOE	1, <u>2</u>	<u>3</u> , <u>4</u>

Now read the payoffs as **cardinal** (outcome utilities)

Pure NE: (B,B) and (H,H)

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	HangOE	1, 2	3, 4

Now read the payoffs as **cardinal** (outcome utilities)

Pure NE: _____

Is there also a mixed NE?

Sunday Morning: The Mixed NE

$q = 1/4$ Blair
 $1 - q = 3/4$
 $p = 1/2$ Alex
 $1 - p = 1/2$

	BuckyDs	HangOE
BuckyDs	4, 3 $\frac{1}{8}$	2, 1 $\frac{3}{8}$
HangOE	1, 2 $\frac{1}{8}$	3, 4 $\frac{3}{8}$

Blair's q makes Alex indifferent:

$$B: 4q + 2(1-q) = 2 + 2q$$

$$H: 1q + 3(1-q) = 3 - 2q$$

$$2 + 2q = 3 - 2q$$

$$4q = 1$$

$$q = 1/4$$

Alex's p makes Blair indifferent:

$$B: 3p + (1-p)2 = 2 + p$$

$$H: 1p + (1-p)4 = 4 - 3p$$

$$2 + p = 4 - 3p$$

$$4p = 2$$

$$p = 1/2$$

Payoffs at the mixed NE: Alex $(\frac{1}{2}, \frac{1}{2})$ Blair $(\frac{1}{4}, \frac{3}{4})$ Better or worse than the pure NE?

$$(B, B) \Rightarrow (4, 3)$$

$$(H, H) \Rightarrow (3, 4)$$

$$MSNE \Rightarrow (2.5, 2.5)$$

MSNE payoff

$$\text{Alex: } \frac{1}{8}(4) + (\frac{3}{8})2 + (\frac{1}{8})1 + \frac{3}{8}(3)$$

$$\frac{4+6+1+9}{8} = \frac{20}{8} = 2.5$$

$$\text{Blair: } \frac{1}{8}(3) + \frac{3}{8}(1) + \frac{1}{8}(2) + \frac{3}{8}(4) =$$

$$\frac{3+3+2+12}{8} = \frac{20}{8} = 2.5$$

Sunday Morning: The Mixed NE

		Blair	
		BuckyDs	HangOE
Alex	BuckyDs	4, 3	2, 1
	HangOE	1, 2	3, 4

Blair's q makes Alex indifferent:

Alex's p makes Blair indifferent:

Payoffs at the mixed NE: Alex _____ Blair _____ Better or worse than the pure NE?

Fun Fact: almost every game has an odd number of NE.

Indifference Isn't Enough

		$\frac{1}{2}$ Player 2 $\frac{1}{2}$		
		D	E	
$\frac{1}{2}$	A	3, 0	0, 2	$\rightarrow \frac{1}{2}3 + \frac{1}{2}0 = 1.5$
$\frac{1}{2}$	B	0, 2	3, 0	$\rightarrow \frac{1}{2}0 + \frac{1}{2}3 = 1.5$
0	C	2, 0	2, 1	$\rightarrow 2$ B.R.

indiff ✓

Try: P1 plays $\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$, P2 plays $\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

P1 indifferent between A and B? $\Pi_1(A) = 1.5$ $\Pi_1(B) = 1.5$ same but not B.R.

Indifference Isn't Enough

		Player 2	
		D	E
Player 1	A	3, 0	0, 2
	B	0, 2	3, 0
	C	2, 0	2, 1

Try: P1 plays $\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$, P2 plays $\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

P1 indifferent between A and B? $\Pi_1(A) =$ $\Pi_1(B) =$

But $\Pi_1(C) =$ 2 So mixing 50-50 over A and B is **not** a NE!

Lesson: Need indifference **and** no better strategy

Best Replies With Three Strategies

Handwritten: q_D Player 2 $1 - q_D$

		D	E
Player 1	A	<u>3</u> , 0	0, <u>2</u>
	B	0, <u>2</u>	<u>3</u> , 0
	C	2, 0	2, <u>1</u>

Handwritten: p_A , p_B , p_C

$$\Pr(D) = q_D$$

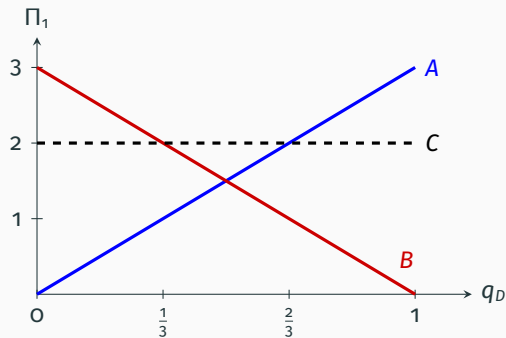
$$\Pr(E) = q_E = 1 - q_D$$

$$\Pi_1(A, q_D) = 3q_D$$

$$\Pi_1(B, q_D) = 3(1 - q_D) = 3 - 3q_D$$

$$\Pi_1(C, q_D) = 2$$

$$BR_1(q_D):$$



Best Replies With Three Strategies

q_D Player 2 $1 - q_D$

	D	E
Player 1 A	3, 0	0, 2
B	0, 2	3, 0
C	2, 0	2, 1

$3q_D + 0(1 - q_D) = 3q_D$
 $0q_D + 3(1 - q_D) = 3 - 3q_D$

$$\Pr(D) = q_D$$

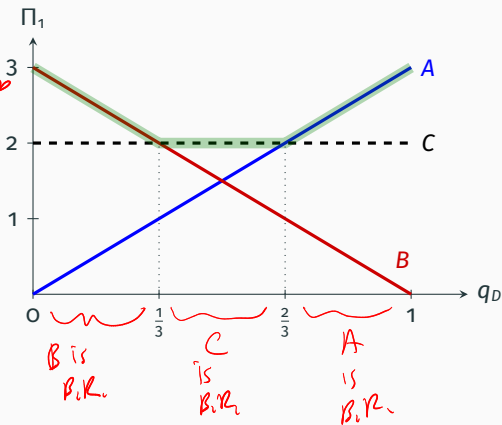
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$$\Pi_1(C, q_D) = 2$$

$BR_1(q_D)$: B if $q_D < \frac{1}{3}$ C if $\frac{1}{3} < q_D < \frac{2}{3}$ A if $q_D > \frac{2}{3}$



Best Replies With Three Strategies

		Player 2	
		D	E
Player 1	A	3, 0	0, 2
	B	0, 2	3, 0
	C	2, 0	2, 1

$$\Pr(D) = q_D$$

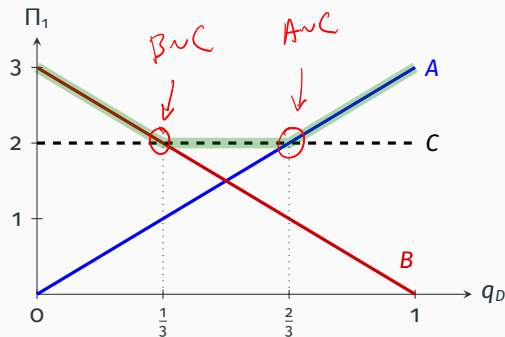
$$\Pr(E) = q_E = 1 - q_D$$

$$\Pi_1(A, q_D) = 3q_D$$

$$\Pi_1(B, q_D) = 3(1 - q_D) = 3 - 3q_D$$

$$\Pi_1(C, q_D) = 2$$

$$BR_1(q_D): \quad B \text{ if } q_D < \frac{1}{3} \quad C \text{ if } \frac{1}{3} < q_D < \frac{2}{3} \quad A \text{ if } q_D > \frac{2}{3}$$



No pure strategy NE $\Rightarrow$ Must be mixing $\Rightarrow$ Must have $q_D = 1/3$ (mix B and C) or $q_D = 2/3$ (mix A and C)

Best Replies With Three Strategies

		Player 2	
		D	E
Player 1	A	3, 0	0, 2
	B	0, 2	3, 0
	C	2, 0	2, 1

$$\Pr(A) = p_A$$

$$\Pr(B) = p_B$$

$$\Pr(C) = p_C = 1 - p_A - p_B$$

$$\Pi_2(D, (p_A, p_B, p_C)) = 2p_B$$

$$\Pi_2(E, (p_A, p_B, p_C)) = 2p_A + 1p_C =$$

$$2p_A + (1 - p_A - p_B) = 1 + p_A - p_B$$

P2 will mix when

$$2p_B = 1 + p_A - p_B$$

$$3p_B = 1 + p_A$$

Can't be true if P1 is mixing between A & C (since $p_B = 0$). So P1 must be mixing between B & C (setting $p_A = 0$). Indifference principle:

$$\Pi_2(D, (p_A, p_B, p_C)) = \Pi_2(E, (p_A, p_B, p_C))$$

$$3p_B = 1 + p_A$$

$$3p_B = 1 + 0$$

$$p_B = \frac{1}{3}$$

Best Replies With Three Strategies

		Player 2	
		D	E
Player 1	A	3, 0	0, 2
	B	0, 2	3, 0
	C	2, 0	2, 1

Mixed-Strategy Nash equilibrium:

P1: $(p_A, p_B, p_C) = (0, 1/3, 2/3)$ (to make P2 indiff. between D and E)

P2: $(q_D, q_E) = (1/3, 2/3)$ (to make P1 indiff. between B and C)

What Does Mixing Mean?

At a mixed NE you're **indifferent**. So why mix exactly $1/3$???

Three readings:

1. **Literal randomizing.** Coaches, kickers, poker players
2. **Beliefs.** σ_2 is what P1 *believes* P2 will do
 - P2 may not randomize at all. P1 just can't predict them
3. **Populations.** 60% of coaches run, 40% pass

Bonanno calls mixed NE “conceptually problematic.” Agree?

Mixing in the Real World

Lots of situations boil down to “be unpredictable”:

- **Penalty kicks.** Kick left or right? Dive left or right?
- **Tennis serves.** To the forehand or the backhand?
- **Play-calling, pitch selection.** Run/pass. Fastball/curveball
- **Poker.** Bluff with a bad hand, or fold?
- **Inspections.** Tax audits, parking tickets, drug tests, fare checks
- **Security.** LAX randomizes its checkpoints and patrols

Lab/field result: Pro soccer and tennis players mix close to NE rates

Not in the book. Sources: Walker & Wooders (2001, *AER*); Chiappori, Levitt & Groseclose (2002, *AER*); Palacios-Huerta (2003, *REStud*); Pita et al. (2008, *AAMAS*).

But Those Are More Complicated

Real versions add:

- More than two strategies (left, middle, right)
- Repeated play: patterns can be spotted and exploited
- Players differ in skill (a left-footed kicker)
- Payoffs nobody knows exactly

Core idea survives: if you're predictable, you get exploited

Dominance, Revisited

Remember This One?

From Chapter 2: M is never a best reply to a **pure** strategy, but isn't dominated

		Player 2		
		L	C	R
Player 1	U	4, 1	2, 0	4, 2
	M	3, 2	3, 1	3, 0
	D	0, 0	4, 2	0, 1

Suppose P2 plays $\begin{pmatrix} L & C & R \\ 0.4 & 0.6 & 0 \end{pmatrix}$

$$\Pi_1(U) =$$

$$\Pi_1(M) =$$

$$\Pi_1(D) =$$

Remember This One?

From Chapter 2: M is never a best reply to a **pure** strategy, but isn't dominated

		Player 2		
		L	C	R
Player 1	U	4, 1	2, 0	4, 2
	M	3, 2	3, 1	3, 0
	D	0, 0	4, 2	0, 1

Suppose P2 plays $\begin{pmatrix} L & C & R \\ 0.4 & 0.6 & 0 \end{pmatrix}$

$$\Pi_1(U) =$$

$$\Pi_1(M) =$$

$$\Pi_1(D) =$$

M is a best reply, to a **mixed** strategy!

Dominated by a Mix

		Player 2	
		D	E
Player 1	A	0, 1	4, 0
	B	1, 2	1, 4
	C	2, 0	0, 1

Is *B* dominated by *A*? By *C*?

Try the mix $\begin{pmatrix} A & B & C \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$: vs. *D*: vs. *E*:

Dominated by a Mix

		Player 2	
		D	E
Player 1	A	0, 1	4, 0
	B	1, 2	1, 4
	C	2, 0	0, 1

Is *B* dominated by *A*? By *C*?

Try the mix $\begin{pmatrix} A & B & C \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$: vs. *D*: vs. *E*:

B is **strictly dominated by a mixed strategy**

So *B* is never a best reply, to anything P2 might do

Pearce's Theorem

Theorem (Pearce, 1984)

In a two-player game, s_i is never a best reply to any mixed strategy of the opponent



s_i is strictly dominated by some mixed strategy of player i

- Fixes Chapter 2's gap: "never a best reply" = "dominated by a mix"
- Opponent's mix = my **belief** about them
 - Dominated by a mix $\iff$ irrational under **every** belief

Three or more players: true if beliefs may be correlated (Bonanno §6.4).

Cardinal IDSDS

Same as IDSDS, but also delete strategies dominated by **mixes**

- Survivors are called **rationalizable** strategies

		Player 2		
		D	E	F
Player 1	A	3, 4	2, 1	1, 2
	B	0, 0	1, 3	4, 1
	C	1, 4	1, 4	2, 6

Step 1:

Step 2:

Step 3:

Step 4:

Careful: needs players to know each other's **cardinal** payoffs. Realistic?

Let's Work One Out

Gameday Parking

Jordan drives to campus for the Michigan game. Three options:

- **Pay** for a gameday lot
- Park illegally in a **West** Campus lot
- Park illegally on **Neil** Ave

CampusParc has one tow crew. It can patrol West Campus or Neil Ave

		CampusParc	
		Patrol West	Patrol Neil
Jordan	Pay	1, 0	1, 0
	West	0, 2	4, 0
	Neil	3, 0	0, 1

Jordan picks the row, CampusParc the column. Payoffs: Jordan, CampusParc. Hypothetical numbers.

1. Is any strategy strictly dominated by a **pure** strategy?
2. Is any strategy strictly dominated by a **mixed** strategy? Delete it
3. Is there a pure-strategy NE?
4. Find the mixed NE and both players' expected payoffs.
Check that Jordan really doesn't want to Pay
5. CampusParc now earns 4 (not 2) for a tow at West Campus.
Whose mix changes? Does Jordan's payoff change?

What Have We Learned, Charlie Brown?

- Lottery in a cell? Its expected utility is the payoff
- **Mixed strategy**: a lottery over your own pure strategies
- $\Pi_i(\sigma)$: weight each cell's payoff by its probability
 - Needs **cardinal** payoffs. That's why we did Chapter 5
- **Mixed NE**: every mix is a best reply to the others
- **Nash (1951)**: every finite game has one
- **Indifference principle**: my mix makes *you* indifferent
 - Necessary, not sufficient. Check unused strategies
- **Pearce**: never a best reply $\iff$ dominated by a mix

Next: mixing in game trees (behavioral strategies)

- Bonanno, *Game Theory*, Chapter 6
 - §6.1: skim. Lotteries in cells, done formally
 - §6.2–6.4: read carefully
- Exercises: §6.5.2 (6.3, 6.4), §6.5.3 (6.7, 6.8, 6.10), §6.5.4 (6.12)
 - 6.13 and 6.14 are harder. 6.15 (the dog in the river) is a challenge
- Solutions in §6.6. Try them first