Chapter 4: General Dynamic Games

ECON 5001
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Based on: *Game Theory* by Giacomo Bonanno

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What Chapter 3 Assumed

Last time, every player knew the entire history when it was her turn.

That ruled out almost everything interesting:

- sealed bids
- a firm choosing capacity without seeing its rival's choice
- poker
- any move made **at the same time** as somebody else's

In this lecture we drop that assumption, then rebuild backward induction so it still works.

Log In: Startup or Safe Job?

- You and a friend have an idea for an ambitious startup
- It only works if **both** of you quit and go all in
- The alternative is a cushy corporate job: safe, decent money
- Go all in alone and the startup fails. You get nothing.

You two decide **at the same time**, with no chance to talk first.

Pick one.

What We Did

		Your friend	
		Startup	Corp job
You	Startup	3, 3	0, 1
	Corp job	1, 0	1, 1

Share who went all in: ____________

Share who took the job: ____________

Two Nash equilibria: (Startup, Startup) and (Corp job, Corp job). One is better for everyone. Getting there requires trusting your friend.

Information Sets

A Familiar Situation

To discourage copying, I print two versions of the exam, white paper and pink, and hand them out so neighbors never share one.

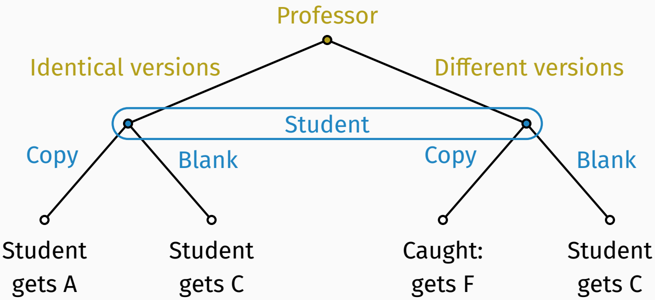
Or do I? Writing genuinely different versions is extra work for me. I might be bluffing: same questions, different paper.

A student who didn't study is sitting next to someone who did.

- Turn in a blank exam and, given earlier grades, he gets a **C**
- Copy, and if the versions are identical he gets an **A**
- Copy, and if the versions differ he is caught and gets an **F**

The moves are sequential. I decide first, but he doesn't *see* my decision.

Drawing What He Doesn't Know



The enclosure is an **information set**: the Student knows he is at one of those two nodes, but not which one.

Definition: Extensive-Form Game-Frame

Everything from Chapter 3, plus one item:

- a finite rooted directed tree
- players $I = \{1, \ldots, n\}$, one assigned to every decision node
- actions on edges, no two edges out of a node sharing an action
- outcomes assigned to terminal nodes

$\star$ For every player i, a **partition** $\mathcal{D}_i$ of her decision nodes D_i. Each cell of the partition is an **information set**.

We write H for a single information set, so $H \in \mathcal{D}_i$.

Restriction: any two nodes in the same information set must offer *exactly the same actions*.

Why must that hold?

A Note on Notation

Three different things, easily confused:

- D_i is the set of **all** decision nodes belonging to player i
- $\mathcal{D}_i$ is the **partition** of D_i, a collection of sets
- $H \in \mathcal{D}_i$ is **one** information set, a single cell

Bonanno writes D and D' where we write H and H'. Watch out: in Chapter 3 he also used D for the set of *all* decision nodes in the tree, so the same letter does two jobs. We'll stick with H for information sets.

Perfect Information Is the Special Case

Every information set is a singleton

$$\Longleftrightarrow$$

Perfect information

So Chapter 3 was not a different theory. It was this one, with the partition as fine as it can be.

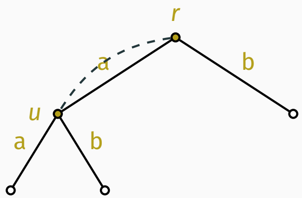
Convention from here: we draw an enclosure **only** around information sets with two or more nodes. A lone node is its own information set.

Perfect Recall: First Condition

Almost all of game theory assumes players don't forget.

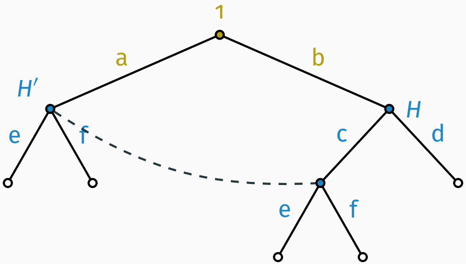
Condition 1
No node in an information set is a **predecessor** of another node in it.

Below, $H = \{r, u\}$ violates it: r comes before u, so Player 1 cannot tell whether she has already moved.

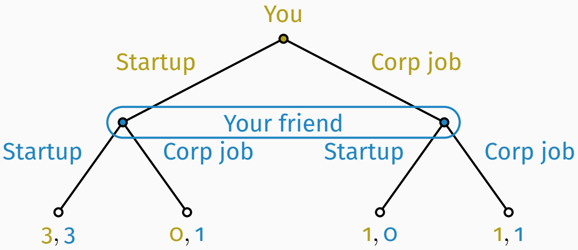


Condition 2
If $x \in H$ leads to $y \in H'$ by action a, then **every** node in H' has a predecessor in H reached by that same a.

Below, H' is Player 2's lower information set. Its right node is reached via H, but its left node never passes through H at all. Player 2 cannot tell whether this is her first move or her second.

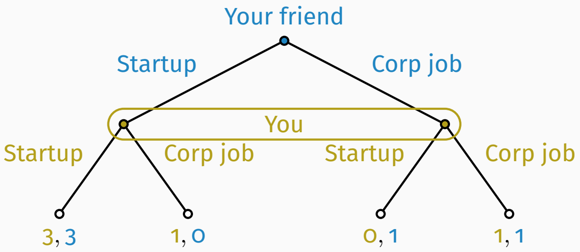


The Same Game, Drawn as a Tree



One of you is drawn as moving first, but the other's information set means she sees nothing. So this is **the same game** as the matrix.

The Same Game, Drawn as a Tree



One of you is drawn as moving first, but the other's information set means she sees nothing. So this is **the same game** as the matrix.

Swap who moves first and nothing changes. Simultaneity is not about time. It is about **what you know when you choose**.

Strategies, Again

One Choice per Information Set

Definition
A **strategy** for a player in an extensive-form game is a list of choices, one for every **information set** of that player.

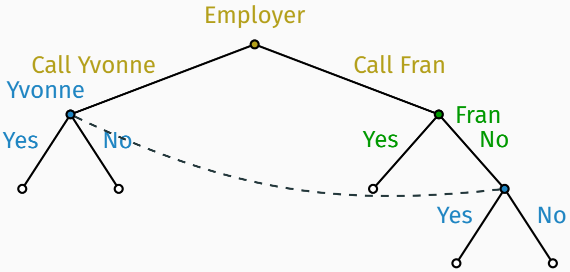
Chapter 3 said “one for every decision node.” That was the same rule, because every node was its own information set.

The change matters: a player can no longer make a plan that depends on something she cannot observe.

A Plan You Are Not Allowed to Make

Two candidates, Yvonne and Fran, interview for one position. The employer says: “don’t call me, I’ll call you.” If the first candidate turns him down, he immediately calls the other, never saying it’s a second offer.

Yvonne would love to plan: *If he calls me first, I say Yes. If I’m the recycled offer, I say No.*



Must choose the **same action at every node** of an information set.

From Tree to Table, Again

Every strategy profile still determines a path, hence an outcome, hence payoffs. So every extensive-form game still has a **strategic form**.

And we could just find its Nash equilibria and stop.

But Chapter 3 taught us not to. Some Nash equilibria of the strategic form rest on threats nobody would carry out.

We fixed that with backward induction, which needs a *last* decision node.

What is the analogue when players move blind?

Subgames

The Idea

A **subgame** is a piece of the tree that could stand alone as a game.

Informal procedure:

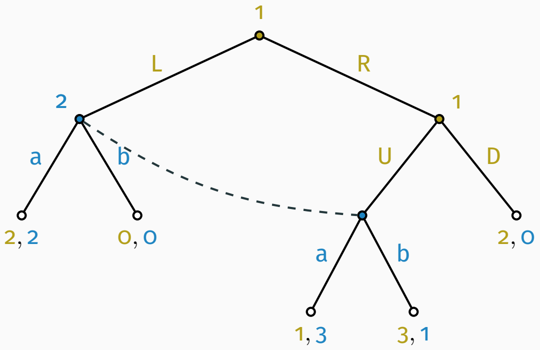
1. Pick a decision node x whose information set is **just** $\{x\}$
2. Draw an oval around x and everything that follows it

If that oval **cuts through** an information set, the oval is illegal. So only call it a subgame if it cuts **no** information set.

"Cutting" means some information set has one node inside your oval and another node outside it.

Every game is a subgame of itself. Anything smaller is a **proper** subgame.

Why Cutting Is Fatal



Player 1's node after *R* looks like a fine place to start. But the oval around it would contain *one* of Player 2's two nodes.

Player 2 can't tell those nodes apart, so you cannot hand her the right-hand piece and call it a game. Half her uncertainty is outside it.

Definition: Subgame

Let $S(x)$ be the set containing x and all its successors.

$S(x)$ is a **subgame starting at** x if:

1. the information set containing x is the singleton $\{x\}$, and
2. for every information set H in the game, if $H \cap S(x) \neq \emptyset$ then $H \subset S(x)$

It is a **proper** subgame if x is not the root.

A subgame is **minimal** if it contains no other subgame strictly inside it. Those are the ones we solve first.

Subgame-Perfect Equilibrium

Let s be a strategy profile for the whole game, and G a subgame. Write $s|_G$ for the **restriction** of s to G: the part of s that prescribes choices at G's information sets, and only those.

Definition
s is a **subgame-perfect equilibrium** if, for **every** subgame G, the restriction $s|_G$ is a Nash equilibrium of G.

The whole game is one of its own subgames, so every subgame-perfect equilibrium is a Nash equilibrium.

Not every Nash equilibrium is subgame perfect. That's the point.

IKEA and Wayfair, Again

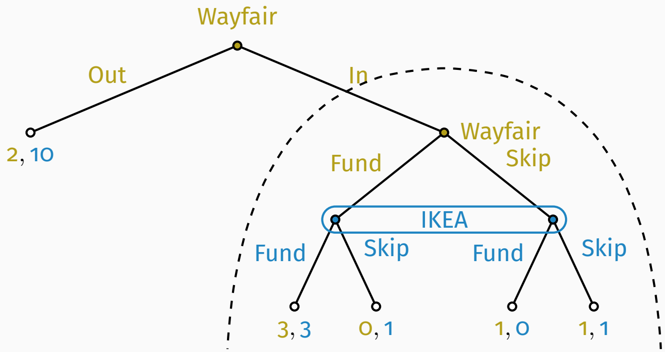
IKEA is the established furniture store at Polaris, earning $10 million a year in profit. Wayfair decides whether to open across I-71.

If Wayfair opens, the two stores **simultaneously** decide whether to fund a joint campaign advertising Polaris as a furniture destination.

- Both fund it: shoppers drive up from all over. Both gain.
- Neither funds it: nothing changes.
- One funds it alone: too small to work, and the money is gone.

That second stage is the startup game you played earlier.

The Whole Game



Annual profit in $millions. First number Wayfair, second IKEA. The circle marks the only proper subgame.

The Algorithm

1. Find a **minimal** subgame.

2. Solve it. Find its Nash equilibria, pick one, and note the strategies.

3. **Mark** the subgame's starting node with that equilibrium's payoff vector, exactly as in backward induction, then delete everything below it. You now have a smaller game.

4. Repeat until no proper subgames remain. Solve what's left. Patch together everything you noted along the way.

Same marking idea as Chapter 3. The only change: a minimal subgame may be a whole simultaneous game rather than a single node, so we mark it with a Nash equilibrium payoff instead of one player's best choice.

If a subgame had several Nash equilibria, run the procedure again with a different one for a different subgame-perfect equilibrium.

If some subgame has *no* Nash equilibrium, neither does the game.

Step 1: Solve the Subgame

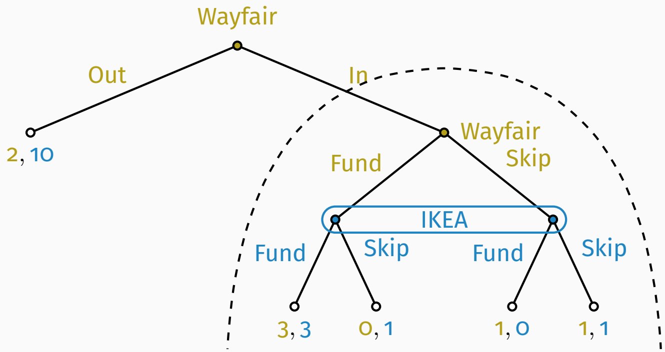
Wayfair \ IKEA	Fund	Skip
Fund	$\underline{3}, \underline{3}$	$0, 1$
Skip	$1, 0$	$\underline{1}, \underline{1}$

Two Nash equilibria: (Fund, Fund) with payoffs $(3, 3)$, and (Skip, Skip) with payoffs $(1, 1)$.

So we have to run the algorithm **twice**.

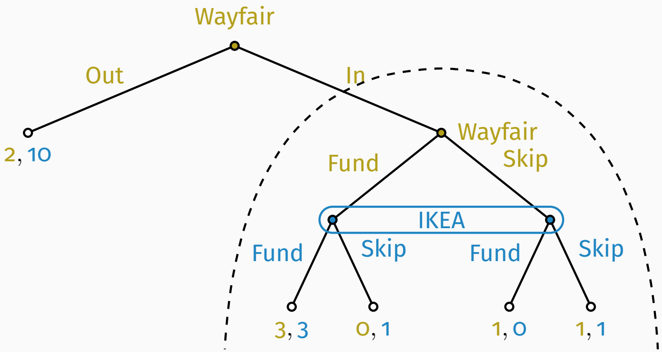
Step 2: Replace and Solve

Using the **(Fund, Fund)** equilibrium.



Step 2: Replace and Solve

Using the **(Skip, Skip)** equilibrium.



Two Subgame-Perfect Equilibria

$$\big((\text{In}, \text{Fund}),\ \text{Fund}\big) \qquad \text{and} \qquad \big((\text{Out}, \text{Skip}),\ \text{Skip}\big)$$

In the second one Wayfair never opens, *because* it expects no campaign.

Nothing about the market changed. The pessimism is self-confirming.

- Both firms would prefer the first equilibrium
- Neither can get there by acting alone
- Subgame perfection does **not** pick between them

Refinements narrow the set of predictions. Not always to one.

The Strategic Form

Wayfair's strategy is a pair: what to do at the root, and what to do in the campaign stage.

Wayfair \ IKEA	Fund	Skip
(Out Fund)	2, 10	2, 10
(Out Skip)	2, 10	2, 10
(In Fund)	3, 3	0, 1
(In Skip)	1, 0	1, 1

The Strategic Form

Wayfair's strategy is a pair: what to do at the root, and what to do in the campaign stage.

		IKEA	
		Fund	Skip
Wayfair	(Out Fund)	$2, \underline{10}$	$\underline{2}, \underline{10}$
	(Out Skip)	$2, \underline{10}$	$\underline{2}, \underline{10}$
	(In Fund)	$\underline{3}, \underline{3}$	$0, 1$
	(In Skip)	$1, 0$	$1, \underline{1}$

Three cells have both payoffs underlined. Three Nash equilibria.

The Nash Equilibrium We Threw Out

The strategic form has **three** Nash equilibria. Wayfair's strategy is a pair: what to do at the root, and what to do in the campaign stage.

Profile	Subgame play	Subgame perfect?
$((\text{In}, \text{Fund}), \text{Fund})$	(Fund, Fund)	yes
$((\text{Out}, \text{Skip}), \text{Skip})$	(Skip, Skip)	yes
$((\text{Out}, \text{Fund}), \text{Skip})$	(Fund, Skip)	**no**

The third is a Nash equilibrium of the whole game. Wayfair staying out is a best reply if it believes IKEA will skip.

But its plan for the subgame is incoherent: if entry happened, funding alone earns 0, and it would rather skip.

Three Things Worth Remembering

- In a **perfect-information** game, subgame-perfect equilibrium is exactly backward induction. Every decision node starts a subgame, so the algorithm walks the tree from the leaves.
- If a game has **no proper subgames**, every Nash equilibrium is subgame perfect. The refinement has nothing to bite on.
- Otherwise subgame perfection is a strict refinement: fewer predictions, all of them Nash.

Bonanno's Figures 4.8 and 4.9 work through larger three-player trees with several subgames. Worth doing before the problem set.

Chance Moves

When Nobody Chose It

Sometimes what happens isn't anyone's decision. A card is dealt, a part fails, a customer walks in.

We model this with a fictitious player: **Nature** (or **Chance**).

- Nature moves at some nodes
- Each of Nature's actions carries a **probability**
- Nature has no payoffs and no objectives, so it isn't really a player

Everything else (information sets, strategies, subgames) is unchanged.

Three Cards

Three cards: one black, two red. Shuffled, face down.

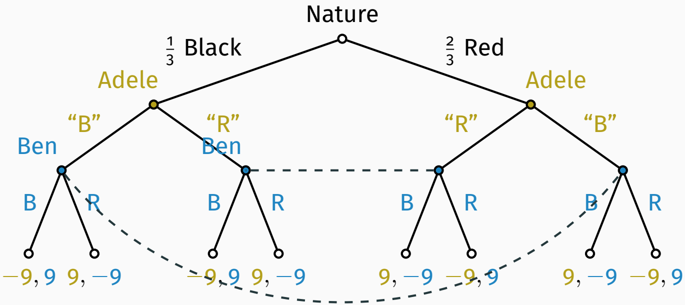
Nature moves first, turning up the top card: black with probability $\frac{1}{3}$, red with probability $\frac{2}{3}$.

Then the players move:

- Adele looks at the top card without showing Ben
- She tells Ben "black" or "red", truthfully or not
- Ben then guesses the true color
- Correct guess: Adele pays Ben \$9. Wrong: Ben pays Adele \$9.

Ben has two information sets, one for each thing Adele might say, and in each he is unsure what the card actually is.

The Card Game



Nature moves first. Adele's label is what she *says*; Ben's is what he *guesses*. First payoff Adele, second Ben.

Ben has two information sets, one for each thing Adele might say. In each, he knows what he heard but not what the card is.

A New Difficulty

Take the profile: Adele always tells the truth, Ben always guesses black.

What's the outcome? It depends on the card, so the outcome is itself a gamble:

Outcome	Adele pays \$9	Ben pays \$9
Probability	$\frac{1}{3}$	$\frac{2}{3}$

We call this a **lottery**. To build a strategic form we need to know how players rank *lotteries*, not just outcomes.

Chapter 5 answers that properly. For now, one shortcut.

Expected Value

Definition

The **expected value** of the money lottery $\begin{pmatrix} \$x_1 & \$x_2 & \cdots & \$x_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$ is

$$x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

$\begin{pmatrix} \$5 & \$15 & \$25 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$ has expected value ______________

$\begin{pmatrix} -\$9 & \$9 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ has expected value ______________

Risk Neutrality

Definition
A player is **risk neutral** if she considers a money lottery to be *just as good as* its expected value.

So a risk-neutral player ranks lotteries by their expected value, and we can use that number as her payoff.

$$L_1 = \begin{pmatrix} \$5 & \$15 & \$25 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix} \qquad L_2 = \begin{pmatrix} \$16 \\ 1 \end{pmatrix} \qquad L_3 = \begin{pmatrix} \$0 & \$32 & \$48 \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

A risk-neutral player ranks these: ______________________________

This is a convenience, not a claim about people. Risk neutrality lets us finish Chapter 4 without the machinery of Chapter 5.

Why the Assumption Is Doing Work

L_2 and L_3 have the same expected value, \$16.

- L_2 pays \$16 for certain
- L_3 pays nothing five times out of eight, and sometimes \$48

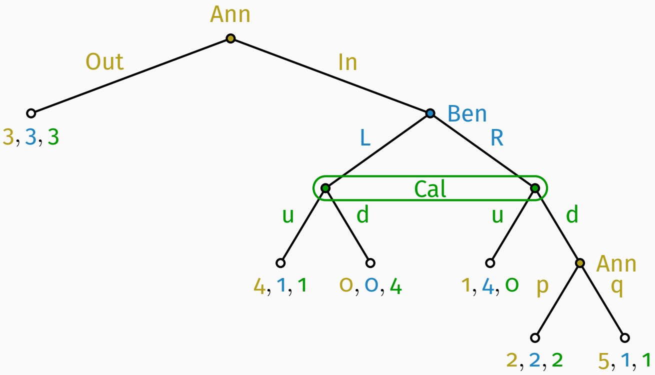
A risk-neutral player is **indifferent** between them.

Most people are not. Ask yourself which you'd take, and notice that your answer is information the expected value throws away.

Recovering it is exactly what Chapter 5 is for.

Let’s Work One Out

A Three-Player Dynamic Game



A Three-Player Dynamic Game

Its strategic form. Ann's strategy is a pair: the root, then her later node.

Cal: u

Ann \ Ben	L	R
(Out p)	3, 3, 3	3, 3, 3
(Out q)	3, 3, 3	3, 3, 3
(In p)	4, 1, 1	1, 4, 0
(In q)	4, 1, 1	1, 4, 0

Cal: d

Ann \ Ben	L	R
(Out p)	3, 3, 3	3, 3, 3
(Out q)	3, 3, 3	3, 3, 3
(In p)	0, 0, 4	2, 2, 2
(In q)	0, 0, 4	5, 1, 1

A Three-Player Dynamic Game

1. List every subgame. Why is there none starting at Cal's left node?
2. How many strategies does each player have?
3. Run the subgame-perfect equilibrium algorithm. Solve the smallest subgame first.
4. Find a Nash equilibrium of the whole game that is **not** subgame perfect.

What Have We Learned, Charlie Brown?

- An **information set** collects nodes a player cannot tell apart; perfect information is the case where all are singletons
- A **strategy** gives one choice per information set, not per node
- A **subgame** starts at a singleton information set and cuts no information set
- **Subgame-perfect equilibrium**: Nash in every subgame. Solve minimal subgames first, replace, repeat
- **Nature** handles chance; risk neutrality ranks lotteries by mean

Next: what if players aren't risk neutral?

Reading & Practice

- Bonanno, *Game Theory*, Chapter 4
- Exercises: §4.6.1 (imperfect information), §4.6.2 (strategies), §4.6.3 (subgames), §4.6.4 (subgame-perfect equilibrium), §4.6.5 (chance moves)
- Exercise 4.2 is the entry game with a simultaneous second stage, close to the IKEA example, with more to it
- Solutions in §4.7