Chapter 2 (Part 2): Auctions, Iterated Dominance, & Nash Equilibrium

ECON 5001
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Based on: *Game Theory* by Giacomo Bonanno

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Where We Left Off

We can now:

- separate a **frame** from a **game**
- write preferences as a complete, transitive $\succsim_i$
- represent $\succsim_i$ with an ordinal utility function
- compare two strategies of one player: **dominance**

And we found: if you have a dominant strategy, play it.

In this lecture: what if you don't?

Log In: Auction Game

- MobLab: Two auctions. You're bidders.
- You'll be told what the item is worth **to you**
  - That number is yours alone; others have different ones
  - Random from $10 to $100 (equally likely)
- Groups of SIX bidders.
- Submit one sealed bid. Highest bid wins.
  - First game: Winner pays their bid
  - Second game: Winner pays highest *losing* bid (2nd highest bid)
- We'll play it **twice**, under two different rules

Your payoff is your value minus what you pay, or zero if you don't win.

No talking. Bid however you think is smart.

What Did You Just Play?

Round 1, first-price auction: the winner pays *their own bid*.

Round 2, second-price auction: the winner pays *the 2nd highest bid*.

In which round did you bid closer to your true value?
Did you (bid $<$ value) or (bid $>$ value)?

Second-Price Auctions

The Rules

A **second-price** (or **Vickrey**) auction:

- sealed bids: nobody sees anyone else's bid
- highest bidder wins
- the winner pays the **second-highest** bid, not their own
- ties are broken by a rule fixed in advance

Why would a seller ever do this? It looks like giving money away.

An Example: The Game-Day Permit

Suppose OSU auctions off a food truck permit near the stadium for game day.
Dos Hermanos and Pitabilities both want it.

- Dos Hermanos (Player 1): estimates **\$30k** extra profit if they get it
  This is their **value** for the permit. Write $v_1 = 30$ (in thousands)
- Suppose bids must be one of \$10k, \$20k, \$30k, \$40k, \$50k
  So $S_1 = S_2 = \{10, 20, 30, 40, 50\}$. They have 5 possible strategies.
  Write bids as $b_1 \in S_1$ and $b_2 \in S_2$
- **Second-price auction:**highest bid wins, pays *2nd highest bid*
- Ties go to Pitabilities (decided by a coin flip beforehand)

Dos Hermanos's payoff in the Second Price Auction:

$$\pi_1 = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ 0 & \text{if } b_1 \leq b_2 \end{cases}$$

Assume utility = money (selfish & greedy)

Only Dos Hermanos's Payoffs Shown

	Pitabilities				
Dos Hermanos ($v_1 = 30$)	10	20	30	40	50
10	$0, \cdot$	$0, \cdot$	$0, \cdot$	$0, \cdot$	$0, \cdot$
20	$20, \cdot$	$0, \cdot$	$0, \cdot$	$0, \cdot$	$0, \cdot$
30	$20, \cdot$	$10, \cdot$	$0, \cdot$	$0, \cdot$	$0, \cdot$
40	$20, \cdot$	$10, \cdot$	$0, \cdot$	$0, \cdot$	$0, \cdot$
50	$20, \cdot$	$10, \cdot$	$0, \cdot$	$-10, \cdot$	$0, \cdot$

Recall: $\pi_1 = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ 0 & \text{if } b_1 \leq b_2 \end{cases}$ with $v_1 = 30$

Reading the Table

- Bidding **30** (their true value) is weakly dominant
- Bidding **40** is equivalent to bidding 30, also weakly dominant
- Bidding **50** is strictly worse than 30. Where, and why?
- Bidding **20** or **10** is weakly worse. Where, and why?

Vickrey's Theorem

Theorem (Vickrey, 1961). In a second-price auction, if player *i* is selfish and greedy, then bidding their true value ($b_i = v_i$) is a **weakly dominant strategy**.

Remarkable, because:

- it holds no matter how many bidders there are
- it holds no matter what anyone else bids
- you don't need to guess, predict, or model your rivals at all

Truth-telling isn't nice here. It's just optimal.

Why It Works: Two Cases

Let v_1 be my value and b_2 the other bid. Compare bidding v_1 to anything else.

Case 1: $b_2 \leq v_1$. Bidding v_1 wins and pays b_2, for a payoff of $v_1 - b_2 \geq 0$.

- Bidding higher: still win, still pay b_2. ________________________
- Bidding below b_2: lose, payoff 0. ________________________

Case 2: $b_2 > v_1$. Bidding v_1 loses, for a payoff of 0.

- Bidding anything below b_2: still lose. ________________________
- Bidding at or above b_2: win, pay b_2. ________________________

Raising your bid can only ever change *whether* you win, never *what you pay*.

The Fine Print

The theorem assumes the bidder is **selfish and greedy**.

Drop that and it fails. Suppose I also care what *you* pay:

- **Spiteful**: if I lose, I’d rather you paid more
- **Benevolent**: if I lose, I’d rather you paid less

In either case, bidding my value is no longer dominant.

Same mechanism, different preferences, different game. Again.

Back to Your Bids

Under second-price, truthful bidding is weakly dominant.

Under first-price, it is not… winning means paying your own bid, so bidding your value guarantees a payoff of exactly zero. You must **shade** down.

Round 1 (first-price), average bid/value: ____________

Round 2 (second-price), average bid/value: ____________

The Pivotal Mechanism

A Real Headline

Davis Enterprise, January 12, 2001:

> *"By consensus, the Davis City Council agreed Wednesday to order a community-wide public opinion poll to gauge how much Davis residents would be willing to pay for a park tax and a public safety tax."*

Will people answer honestly?

- If my answer doesn't affect my tax bill:

- If my answer does affect my tax bill:

The Setup

A public project such as building the Upper Arlington Community Center costs \$$C$ in total. There are n residents.

- Resident i's share of the cost is c_i, with $c_1 + \cdots + c_n = C$
  (shares are announced in advance and need not be equal)
- Resident i starts with wealth m_i
- If built, resident i gets benefit worth v_i (possibly **negative**)
  (you live on the street it's going on: traffic, lights, parking)

$$U_i(\$m) = \begin{cases} m & \text{if the project is not built} \\ m + v_i & \text{if it is} \end{cases}$$

Efficiency: build it exactly when the project is worth more to the residents together than it costs to build. Social benefit beats total cost.

Why Efficiency Isn't Enough

$n = 2,\ m_1 = 50,\ m_2 = 60,\ v_1 = 19,\ v_2 = -15,\ C = 6,\ c_1 = 6,\ c_2 = 0.$

$\sum v_i = 19 - 15 = 4 < 6 = C$, so it should *not* be built.

	Not built	Built
Resident 1	50	$50 + 19 - 6 = 63$
Resident 2	60	$60 - 15 = 45$

Gain of 13 to one, loss of 15 to the other.

The government would happily follow the rule if it knew v_1 and v_2.

It doesn't. And asking is exactly the problem.

Clarke's Idea

Each resident *i* reports a number w_i, which represents their reported benefit (or harm, if negative).
They may lie: nothing forces $w_i = v_i$.

Decision rule:

$$\text{Build?} \quad \begin{cases} \text{Yes} & \text{if } \sum_{j=1}^{n} w_j > C \\ \text{No} & \text{if } \sum_{j=1}^{n} w_j \leq C \end{cases}$$

Then the twist. Resident *i* is **pivotal** if the decision *without their report* would differ from the decision *with* it.

- Not pivotal $\Rightarrow$ they pay no tax
- Pivotal $\Rightarrow$ they pay $\left| \sum_{j \neq i} w_j - \sum_{j \neq i} c_j \right|$

Worked Example

$n = 3$, $C = 10$, $c_1 = 3$, $c_2 = 2$, $c_3 = 5$. Reports: $w_1 = -1$, $w_2 = 8$, $w_3 = 3$.

$\sum w_j = 10$, which is *not* greater than $C = 10$, so the project is **not built**.

i	$\sum_{j \neq i} w_j$	$\sum_{j \neq i} c_j$	Decision without i	Tax
1	$8 + 3 = 11$	$2 + 5 = 7$	Build (pivotal)	$\lvert 11 - 7 \rvert = 4$
2	$-1 + 3 = 2$	$3 + 5 = 8$	Don't (not pivotal)	0
3	$-1 + 8 = 7$	$3 + 2 = 5$	Build (pivotal)	$\lvert 7 - 5 \rvert = 2$

Note residents 1 and 3 each flipped the outcome, and each pays for it.

“Then I’ll Just Avoid Being Pivotal”

You can guarantee it: report $w_i = c_i$ and you are never pivotal, never taxed.

Is that smart? Take $n = 4$, $C = 15$, $c = (5, 0, 5, 5)$. You are resident 1, with $m_1 = 40$ and $v_1 = 25$. You expect $w_2 = -40$, $w_3 = 15$, $w_4 = 20$.

Report $w_1 = c_1 = 5$**:** $\sum w_j = 0 < 15$, not built, and you pay nothing.

$$U_1 = 40$$

Report truthfully, $w_1 = 25$**:** $\sum w_j = 20 > 15$, so it *is* built. You are pivotal, taxed $|(-5) - (10)| = 15$.

$$U_1 = 40 + 25 - 5 - 15 = 45$$

Clarke’s Theorem

Theorem (Clarke, 1971). In the pivotal mechanism, reporting truthfully ($w_i = v_i$) is a **weakly dominant strategy** for every player.

Same shape as Vickrey: the tax you pay is built out of *other people’s* numbers, so your report moves the decision without moving your own bill.

Intellectual History

Vickrey, 1961

Clarke, 1971

Groves, 1973

- **1973**: Groves solves the public goods problem in general
- **1971**: Clarke had already published the special case
- **1961**: Vickrey had it all along, in a paper about auctions

The “pivotal mechanism” is often called “the **VCG** mechanism.”

Vickrey won the Nobel Prize in 1996 but died three days after it was announced.

Vickrey (1961), *J. Finance*; Clarke (1971), *Public Choice*; Groves (1973), *Econometrica*.

Iterated Deletion

The Problem

If a player has a dominant strategy, they should play it.

But most games don't have one. So what then?

New idea: even without a dominant strategy, a rational player will never play a **strictly dominated** one.

And if I know you're rational, I know you won't either… which shrinks the game I'm actually facing.

And then *that* smaller game may have dominated strategies of its own.

IDSDS

Iterated Deletion of Strictly Dominated Strategies.

Start with game G.

- G^1: delete every strictly dominated strategy, for every player
- G^2: do it again, inside G^1
- $\vdots$
- G^∞: the output, reached in finitely many steps if G is finite

If G^∞ is a single strategy profile, we call it the **iterated strict dominant-strategy profile**. Otherwise it's just "the output of IDSDS."

Order doesn't matter: any sequence of deletions of strictly dominated strategies reaches the same output.

Doing It

Player 2

		L	C	R
Player 1	T	4, 3	5, 1	6, 2
	M	2, 1	8, 4	3, 6
	B	3, 0	9, 6	2, 8

Nobody has a dominant strategy here. Check a couple:

- T vs. M for Player 1: $4 > 2$, but $5 < 8$ $\Rightarrow$ neither dominates
- T vs. B for Player 1: $6 > 2$, but $5 < 9$ $\Rightarrow$ neither dominates

So start with Player 2.

Doing It

Step 1. For Player 2, *R* strictly dominates *C*: $2 > 1$, $6 > 4$, $8 > 6$. Delete *C*.

		Player 2	
		L	R
Player 1	T	4, 3	6, 2
	M	2, 1	3, 6
	B	3, 0	2, 8

Step 2. Now *T* strictly dominates *both M* and *B* for Player 1, which it did not do a moment ago. Delete them.

Doing It

Player 2

		L	R
Player 1	T	4, 3	6, 2

Step 3. Player 2 now faces a one-row game: *L* gives 3, *R* gives 2. Delete *R*.

$$G^{\infty} = \{(T, L)\}$$

Nobody had a dominant strategy at the start, and yet the game has a unique prediction, *provided* each player knows the other is rational, and knows that the other knows, and so on.

Bonanno's Example

Four rounds instead of three.

		Player 2			
		e	f	g	h
Player 1	A	6, 3	4, 4	4, 1	3, 0
	B	5, 4	6, 3	0, 2	5, 1
	C	5, 0	3, 2	6, 1	4, 0
	D	2, 0	2, 3	3, 3	6, 1

1. delete h (dominated by g)
2. delete D (dominated by C)
3. delete g (dominated by f)
4. delete C (dominated by A)

Output: $\{(A, e), (A, f), (B, e), (B, f)\}$, not a single profile.

Log In Again: Pick a Number

MobLab: Beauty contest

Choose an integer from **1** to **100**.

Whoever is closest to $\frac{2}{3} \times$(the average of everyone's number) wins.

No talking. We'll play a few rounds.

Results

	Average	**Winning number**
Round 1:	____________	____________
Round 2:	____________	____________
Round 3:	____________	____________

What happened between rounds? And why?

Back to Your Numbers

$\frac{2}{3}$ of the average can never exceed $\frac{2}{3} \times 100 \approx 66.7$.

- So 67 strictly dominates everything above it. Delete 68–100.
- Now the target can't exceed $\frac{2}{3} \times 67 \approx 44.7$. Delete 45–67.
- Now it can't exceed $\frac{2}{3} \times 45 = 30$. Delete 31–45.
- Keep going … where does it end?

So why didn't we all play that in round 1?

IDWDS, and Why It's Trickier

What if we delete **weakly** dominated strategies too?

Player 2

Player 1		L	R
	A	4, 0	0, 0
	T	3, 2	2, 2
	M	1, 1	0, 0
	B	0, 0	1, 1

Two perfectly reasonable places to start:

- M is strictly dominated by T $(3 > 1, 2 > 0)$
- B is strictly dominated by T $(3 > 0, 2 > 1)$

Order One: Delete *M* First

Delete $M \Rightarrow$ for Player 2, R now weakly dominates $L \Rightarrow$ delete L

Player 2

		R
Player 1	A	0, 0
	T	2, 2
	B	1, 1

Now *A* and *B* are strictly dominated by *T*.

Answer: (T, R), payoffs $(2, 2)$

Order Two: Delete *B* First

Delete B $\Rightarrow$ for Player 2, L now weakly dominates R $\Rightarrow$ delete R

		Player 2
		L
Player 1	A	4, 0
	T	3, 2
	M	1, 1

Now T and M are strictly dominated by A.

$$\text{Answer: } (A, L), \text{ payoffs } (4, 0)$$

So the Procedure Isn't Well Defined

Two legitimate orders, two different answers… and not by a little: $(2, 2)$ versus $(4, 0)$.

Fix (Definition). At every step, identify *all* weakly or strictly dominated strategies for *all* players, and delete them **simultaneously**.

Applied to that game, IDWDS gives $\{(A, L), (A, R), (T, L), (T, R)\}$.

Note this is not a single profile, so the game has no iterated weak dominant-strategy solution.

A Word of Caution

IDSDS has a clean story: rationality, plus knowing others are rational, and so on.

IDWDS does not.

- To delete a *weakly* dominated strategy, you must believe your opponent might play anything. Otherwise the tie-breaking case never arises
- But deletion is precisely the act of ruling strategies out

So the justification needs rationality *and* a notion of caution, and the two pull against each other.

We won't resolve that here. Just don't treat the two procedures as equally innocent.

Nash Equilibrium

Still Not Enough

Games solved outright by IDSDS or IDWDS are the exception.

We need something that applies to *every* game.

The idea: stop asking “what will a rational player do?” and start asking

“which profiles are **stable**?”

Definition: Nash Equilibrium (Two Players)

A strategy profile $s^* = (s_1^*, s_2^*)$ is a **Nash equilibrium** if

1. for every $s_1 \in S_1$: $\pi_1(s_1^*, s_2^*) \ \geq \ \pi_1(s_1, s_2^*)$
2. for every $s_2 \in S_2$: $\pi_2(s_1^*, s_2^*) \ \geq \ \pi_2(s_1^*, s_2)$

In words: **holding the other player's strategy fixed**, neither player can do better by changing their own.

Note what's fixed and what varies in each line. That's the whole definition.

John Forbes Nash Jr. (1928–2015)

- Duffin’s reference letter, 1948, in full:
  “He is a mathematical genius.”
- Princeton PhD, 1950: 26 pages, two citations
- One of the two was his own
- Nobel, 1994, with Harsanyi and Selten
- Abel Prize, 2015: PDEs, not games
- Schizophrenia from 1959, three decades away
- Died May 2015, coming home from the Abel ceremony

Finding Them

		Player 2		
		L	C	R
Player 1	T	3, 2	0, 0	1, 1
	M	3, 0	1, 5	4, 4
	B	1, 0	2, 3	3, 0

Check (T, L):

- Player 1, holding L fixed: T gives 3; M gives 3; B gives 1 ✓
- Player 2, holding T fixed: L gives 2; C gives 0; R gives 1 ✓

So (T, L) is a Nash equilibrium. Find the other one.

Four Ways to Say It

No regret. After seeing what the other player did, nobody wishes they had chosen differently.

Self-enforcing agreement. If the players could talk beforehand and shake on s^*, nobody would want to break it, assuming the other keeps it.

Viable recommendation. If a third party publicly recommended s^*, nobody would want to deviate, assuming the others comply.

Rationality with correct beliefs. Each player best-responds to what they believe the other will do, and those beliefs are right.

These are all the same inequalities in different clothes.

Best Replies

Strategy s_i is a **best reply** to s_{-i} if

$$\pi_i(s_i, s_{-i}) \;\geq\; \pi_i(s_i', s_{-i}) \quad \text{for every } s_i' \in S_i$$

Equivalent definition of Nash equilibrium:

$$s \text{ is a Nash equilibrium} \iff \text{every } s_i \text{ is a best reply to } s_{-i}$$

This is the version worth memorizing. It also tells you how to *find* equilibria.

The Underlining Trick

1. In each **column**, underline Player 1's largest payoff (all ties)
2. In each **row**, underline Player 2's largest payoff (all ties)
3. Any cell with **both** numbers underlined is a Nash equilibrium

		Player 2			
		E	F	G	H
Player 1	A	4, 0	3, 2	2, 3	4, 1
	B	4, 2	2, 1	1, 2	0, 2
	C	3, 6	5, 5	3, 1	5, 0
	D	2, 3	3, 2	1, 2	3, 3

Your turn. Columns first, then rows.

The Underlining Trick

Player 2

		E	F	G	H
Player 1	A	$\underline{4}, 0$	$3, 2$	$2, \underline{3}$	$4, 1$
	B	$\underline{4}, \underline{2}$	$2, 1$	$1, \underline{2}$	$0, \underline{2}$
	C	$3, \underline{6}$	$\underline{5}, 5$	$\underline{3}, 1$	$\underline{5}, 0$
	D	$2, \underline{3}$	$3, 2$	$1, 2$	$3, \underline{3}$

Only (B, E) has both payoffs underlined: the unique Nash equilibrium.

Watch the near-misses:

- (A, E): Player 1 is happy, Player 2 would rather move to G
- (C, F): Player 1 is happy, Player 2 would rather move to E
- (D, H): Player 2 is happy, Player 1 would rather move to C

More Than Two Players

A profile $s^* \in S$ is a Nash equilibrium if, for every player i,

$$\pi_i(s^*) \;\geq\; \pi_i(s_1^*, \ldots, s_{i-1}^*,\ s_i,\ s_{i+1}^*, \ldots, s_n^*) \quad \text{for all } s_i \in S_i$$

n inequalities, one per player. Nothing new, just more of them.

Three Players, Two Tables

One table per strategy of Player 3. Each cell is (π_1, π_2, π_3).

- underline Players 1 and 2 as before, *within* each table
- underline Player 3 if it is their largest *across* tables, same cell

		Player 2	
		L	R
Player 1	T	4, 1, 4	0, 0, 1
	B	3, 1, 3	3, 3, 2

Player 3 plays *E*

		Player 2	
		L	R
Player 1	T	0, 0, 1	2, 2, 3
	B	1, 4, 2	1, 0, 3

Player 3 plays *F*

Two equilibria in here. Find them.

Three Players, Two Tables

One table per strategy of Player 3. Each cell is (π_1, π_2, π_3).

- underline Players 1 and 2 as before, *within* each table
- underline Player 3 if it is their largest *across* tables, same cell

Player 3 plays *E*

		Player 2	
		L	R
Player 1	T	4, 1, 4	0, 0, 1
	B	3, 1, 3	3, 3, 2

Player 3 plays *F*

		Player 2	
		L	R
Player 1	T	0, 0, 1	2, 2, 3
	B	1, 4, 2	1, 0, 3

Two equilibria in here. Find them.

(T, L, E) and (T, R, F).

Check (B, R, E) and (B, L, F): Players 1 and 2 are both content, and only Player 3 wants to move. That is the comparison you cannot see inside one table.

Nash Survives IDSDS

All three procedures return a **set of strategy profiles**, not one prediction:

- $NE(G)$: the profiles satisfying the Nash inequalities
- $IDSDS(G)$: the profiles still standing when strict deletion halts
- $IDWDS(G)$: the same, for weak deletion

Each is a subset of S, and each can be empty, a single profile, or many.

Theorem. For every ordinal strategic-form game G,

$$NE(G) \subseteq IDSDS(G)$$

- A Nash equilibrium never uses a strategy that IDSDS deletes
- So IDSDS never throws away an equilibrium: it's safe
- But the same is **not** true of IDWDS: it is possible that $NE(G) \neq \emptyset$ and yet $NE(G) \cap IDWDS(G) = \emptyset$

Strict Nash Equilibrium

s^* is a **strict** Nash equilibrium if every player is *strictly* worse off deviating, rather than merely no better off.

$$\text{Nash:} \quad \pi_i(s^*) \geq \pi_i(s_i, s^*_{-i}) \quad \text{for every } s_i \in S_i$$

$$\text{Strict:} \quad \pi_i(s^*) > \pi_i(s_i, s^*_{-i}) \quad \text{for every } s_i \neq s_i^*$$

Two changes: $\geq$ becomes $>$, and s_i^* itself is excluded. It has to be, since $\pi_i(s^*) > \pi_i(s^*)$ is false.

Theorem. If $IDSDS(G) = \{s^*\}$ then s^* is a strict Nash equilibrium.

If $IDWDS(G) = \{s^*\}$ then s^* is a Nash equilibrium, but not necessarily a strict one.

A Bigger Example

Suppose OSU donor Ratmir Timashev makes the following offer to 50 students. Each secretly writes down a request: a multiple of $10, up to $100.

- If **5 or fewer** students ask for $100, everyone gets what they asked for
- Otherwise, everyone gets nothing

All students are selfish and greedy. What are the Nash equilibria?

A Bigger Example

There are two families of equilibria.

(a) Seven or more students request $100.
Everyone gets nothing. Can anyone gain by switching alone?

(b) Exactly five request $100, and all 45 others request $90.
Everyone is paid. Can anyone gain by switching alone?

Convince yourself that *exactly six* requesting $100 is not an equilibrium.

When There Is None

Matching Pennies. Both match ⇒ Player 1 wins; otherwise Player 2 wins.

		Player 2	
		H	T
Player 1	H	$\underline{1}, 0$	$0, \underline{1}$
	T	$0, \underline{1}$	$\underline{1}, 0$

No cell has both payoffs underlined.

With only ordinal payoffs, a game need not have **any** Nash equilibrium.

Fixing this requires letting players randomize, and randomizing requires cardinal payoffs. That's Chapter 5.

Infinite Strategy Sets

No Table, Same Definition

Two players. Each writes down a real number ≥ 1, so $S_1 = S_2 = [1, \infty)$.

Player 1 writes x, Player 2 writes y:

$$\pi_1(x, y) = \begin{cases} x - 1 & \text{if } x < y \\ 0 & \text{if } x \geq y \end{cases} \qquad \pi_2(x, y) = \begin{cases} y - 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

You want to write the larger… no, the *smaller* number. But not too small.

What are the Nash equilibria?

The Answer Is $(1, 1)$, and Only $(1, 1)$

It is an equilibrium. At $(1, 1)$ both get 0. If Player 1 switches to any $x > 1$, then $x \geq y = 1$, so they still get 0.

Nothing else is.

- If $x = y > 1$: Player 1 gets 0, but any $\hat{x}$ with $1 < \hat{x} < x$ gives $\hat{x} - 1 > 0$
- If $x < y$: Player 1 gets $x - 1$, but any $\hat{x}$ with $x < \hat{x} < y$ gives $\hat{x} - 1 > x - 1$
- If $y < x$: same argument for Player 2

And now the strange part. For Player 1, $x = 1$ is **weakly dominated** by every other strategy.

So the unique Nash equilibrium has both players using a weakly dominated strategy.

Cournot Competition

Shell and Speedway sell an identical good: a gallon of regular. Station *i* chooses quantity $q_i \geq 0$ at constant unit cost c. The price is a *function* of total output:

$$P(q_1, q_2) = a - b(q_1 + q_2), \qquad a > c,\ b > 0$$

Shell's profit:

$$\pi_1(q_1, q_2) = P(q_1, q_2)\, q_1 - cq_1 = (a - c)q_1 - bq_1^2 - bq_1 q_2$$

Strategy sets are $[0, \infty)$ (infinite). But a Nash equilibrium is still just a pair $(\bar{q}_1, \bar{q}_2)$ where neither station wants to move.

Cournot wrote this down in 1838, inventing Nash equilibrium about 120 years early, for this one class of games.

Variables and Parameters

Not every letter in a model plays the same role.

Parameters: a, b, c

- fixed numbers, known to everyone, chosen by nobody
- placeholders for a number that is already settled before anyone moves
- left as letters so one solution covers every version of the game

Choice variables: q_1, q_2

- placeholders for a number that someone gets to pick
- these are the strategies: $S_i = [0, \infty)$

Derived values (functions of the variables): $P(q_1, q_2)$, $\pi_i(q_1, q_2)$

- not fixed in advance, and nobody chooses them directly
- they are *functions*: feed in the choices, get a number out

For any letter, ask: could a player have made it come out differently?

Maximizing, With Two Variables

From the math review: to maximize $f(x)$, set $f'(x) = 0$.

Shell's profit depends on *two* quantities:

$$\pi_1(q_1, q_2) = (a - c)q_1 - bq_1^2 - bq_1q_2$$

But Shell only chooses q_1. Speedway's q_2 is not Shell's to pick.

So treat q_2 as a fixed number and maximize over q_1 alone.

Partial Derivatives

A **partial derivative** is an ordinary derivative taken while every other variable is held constant, treated as though it were just a number.

$$\frac{\partial f}{\partial x} : \quad \text{differentiate in } x, \text{ treat } y \text{ as a constant}$$

The curly ∂ is the only new thing. The rules are the ones you already know.

$f(x, y) = 3x^2 + 5xy + y^3$

$\frac{\partial f}{\partial x} =$ ________________________________

$\frac{\partial f}{\partial y} =$ ________________________________

Why That Works

Same $f(x, y) = 3x^2 + 5xy + y^3$. Suppose y is stuck at 2:

$$f(x, 2) = 3x^2 + 10x + 8$$

That is a function of x alone. Differentiate it the ordinary way:

$\frac{d}{dx} f(x, 2) =$ ____________________________

Now compare it to $\frac{\partial f}{\partial x}$ with $y = 2$ substituted in.

Holding a variable constant and plugging in a constant are the same operation.

Finding the Maximizer

Two conditions, in words first:

- **First order**: at the top, the slope in the *x* direction is flat
- **Second order**: the curve bends downward there, so flat means a peak rather than a valley

$$\underbrace{\frac{\partial f}{\partial x} = 0}_{\text{slope is flat}} \qquad \text{and} \qquad \underbrace{\frac{\partial^2 f}{\partial x^2} < 0}_{\text{bending downward}}$$

Skip the second condition and a flat point could just as easily be a minimum.

Shell’s Problem

$$\pi_1(q_1, q_2) = (a - c)q_1 - bq_1^2 - bq_1q_2$$

Hold q_2 fixed. First order:

$$\frac{\partial \pi_1}{\partial q_1} = \underline{\hspace{8cm}} = 0$$

Second order:

$\frac{\partial^2 \pi_1}{\partial q_1^2} = \underline{\hspace{4.5cm}}$, negative because $b > 0$, so the flat point is a maximum.

From First-Order Condition to Best Reply

Shell's first-order condition holds at Shell's optimum, *whatever* q_2 turns out to be:

$$a - c - 2bq_1 - bq_2 = 0$$

So solve it for q_1:

$$BR_1(q_2) = \frac{a - c - bq_2}{2b}$$

The second-order condition was $-2b < 0$, so this really is Shell's best, not its worst.

A Best Reply Is a Function

$$BR_1(q_2) = \frac{a - c - bq_2}{2b}$$

Feed it a quantity for Speedway, and it hands back Shell's best answer to that quantity. One input, one output.

- If Speedway floods the market, BR_1 tells Shell to pull back
- If Speedway sits out, BR_1 tells Shell to act like a monopolist

Speedway's problem is the same with the labels swapped:

$$BR_2(q_1) = \frac{a - c - bq_1}{2b}$$

$BR_i(q_{-i})$ is truncated at 0 if needed.

Nash Equilibrium Is a Fixed Point

Recall the best-reply version of the definition: (q_1^*, q_2^*) is a Nash equilibrium exactly when each is a best reply to the other.

$$q_1^* = BR_1(q_2^*) \qquad \text{and} \qquad q_2^* = BR_2(q_1^*)$$

Substitute the second into the first:

$$q_1^* = BR_1\big(BR_2(q_1^*)\big)$$

Two equations in two unknowns just became **one** equation in one unknown.

Start at q_1^*, ask what Speedway would do, then ask what Shell would do back. A Nash equilibrium is a quantity that returns to *itself* after that round trip.

Solving the Fixed Point

$$q_1 = \frac{a - c - b\,BR_2(q_1)}{2b}, \qquad BR_2(q_1) = \frac{a - c - bq_1}{2b}$$

Substitute, then solve for q_1:

And Then the Other One

Feed q_1^* back through Speedway's best reply:

$$q_2^* = BR_2(q_1^*) = \frac{a - c - b\left(\frac{a-c}{3b}\right)}{2b}$$

$q_2^* =$ ________________________

No surprise that it matches q_1^*: the two stations face identical problems, so the equilibrium had to be symmetric.

Worth checking directly that $BR_1(q_2^*) = q_1^*$. It does.

Cournot: The Numbers

At $\bar{q}_1 = \bar{q}_2 = \dfrac{a-c}{3b}$:

- Price: $P(\bar{q}_1, \bar{q}_2) = \dfrac{a+2c}{3}$
- Each profit: $\dfrac{(a-c)^2}{9b}$

Try it: $a = 25,\ b = 2,\ c = 1.$

$\bar{q}_1 = \bar{q}_2 =$ ____________ $P =$ ____________ profit $=$ ____________

Let’s Work One Out

Three Bars on High Street

Ann, Ben, and Cal each run a bar. Before a home game each independently commits to a **Big** promotion or a **Small** one.

		Ben	
		Big	Small
Ann	Big	2, 2, 2	0, 3, 1
	Small	3, 0, 1	1, 1, 0

Cal: Big

		Ben	
		Big	Small
Ann	Big	1, 1, 3	0, 0, 2
	Small	0, 2, 0	2, 2, 3

Cal: Small

Ann picks the row, Ben the column, Cal the table. Payoffs in order: Ann, Ben, Cal.

Three Bars on High Street

1. Does anyone have a dominant strategy?
2. Does IDSDS delete anything at all?
3. Find every Nash equilibrium. Underline Ann's best payoff in each column, Ben's in each row, and Cal's whenever it beats her payoff in the same cell of the other table.
4. Is either equilibrium Pareto superior to the other?

What Have We Learned, Charlie Brown?

- **Second-price auction**: truthful bidding is weakly dominant (Vickrey)
- **Pivotal mechanism**: truthful reporting is weakly dominant (Clarke)
- **IDSDS**: delete strictly dominated strategies, repeat (order-free, justified by common knowledge of rationality)
- **IDWDS**: order matters, so delete simultaneously, and the justification is murkier
- **Nash equilibrium**: nobody gains by deviating alone
- $NE \subseteq IDSDS$, but a game may have no Nash equilibrium at all

Next: what happens when players can randomize.

Reading & Practice

- Bonanno, *Game Theory*, §2.3–2.7
- Exercises: §2.9.3 (auctions), §2.9.4 (pivotal mechanism), §2.9.5 (iterated deletion), §2.9.6 (Nash equilibrium), §2.9.7 (infinite games)
- Proofs of both theorems are in §2.8
- Solutions in §2.10. Try them first