Chapter 2 (Part 1): Ordinal Games in Strategic Form

ECON 5001
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Based on: *Game Theory* by Giacomo Bonanno

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First: A Video

Click me: The Golden Balls Video

https://www.facebook.com/SebTalksBoxing/videos/woman-wins-100k-with-sensational-shthousery/1127818344236313/

Golden Balls

A British game show. Two contestants, Sarah and Steven, have built up a jackpot of $100,000.

Each secretly picks one of two balls:

- one says **Split**
- one says **Steal**

They choose **simultaneously**. Then both are revealed.

- Both Split: each gets $50,000
- One Steals: the stealer gets all $100,000, the other gets nothing
- Both Steal: nobody gets anything

Golden Balls: The Table

		Steven	
		Split	Steal
Sarah	Split	Sarah: \$50k Steven: \$50k	Sarah: \$0 Steven: \$100k
	Steal	Sarah: \$100k Steven: \$0	Sarah: \$0 Steven: \$0

Each row is a choice for Sarah, each column a choice for Steven.
Each cell is what **happens** (the “outcome”)

This Game Has a Name

The most studied game in game theory:

Prisoner's Dilemma

Two players, each with a **cooperate** option and a **defect** option, where

- defecting looks better *no matter what* the other player does, and
- if both defect, both end up worse off than if both had cooperated

Split = cooperate. Steal = defect.

The Same Dilemma, Out in the Wild

Messier stories, same skeleton (**cooperate** vs. **defect**):

- **Arms races**: hold your build-up vs. out-arm them — both spend fortunes, neither safer
- **Carbon emissions**: cut vs. free-ride on everyone else's cuts
- **A shared fishery**: fish your quota vs. overfish — the stock collapses for all
- **Doping in sport**: race clean vs. dope — if everyone dopes, the medals go to the same people, at a cost
- **Price wars**: hold your price vs. undercut the rival — margins vanish on both sides
- **Ad spending**: modest campaign vs. outspend the rival — market shares barely move
- **Antibiotics**: prescribe sparingly vs. prescribe on demand — resistance is everyone's problem
- **Attack ads**: stay positive vs. go negative — both get dragged down
- **Standing at a concert**: stay seated vs. stand for a better view — everyone stands, nobody sees more

Theory vs. Reality

The gaps between theory and reality:

- **many** players, not two (fishing fleets, countries, fans in a stadium)
- **degrees** of cooperation, not two buttons (how much to emit, how far to cut price)
- **repetition**, reputation, and the chance to retaliate later
- **imperfect monitoring**: you can't always tell who defected
- **outside enforcement**: treaties, regulators, drug testing, cartel policing

None of that changes the core tension:

what's best for me, taken alone, leaves us all worse off.

Before We Analyze It: Let's Play

- Phone, tablet, or laptop (whatever you've got)
- Log in to **MobLab** and join our session
- Class code: `dv1cf8ft`
- One decision: **Split** or **Steal** (Cooperate or Defect)

No talking, no signaling. Choose what you'd actually choose.

For Fun: What Would the Philosophers Say? (Not on Quiz/Test)

Philosopher	**My (very amateurish) Understanding**
Kant	Categorical imperative: What if *everyone* did that? Thus, $C \succ D$
Hobbes	Life is “solitary, poor, nasty, brutish, and short.” We need a government
Locke	People have a natural duty to each other, even without government
Hume	Cooperate becomes convention through repeated play (ex: two farmers)
Rousseau	Trust and the social contract keep society functioning. Play C
Bentham/Mill	Cooperate because it maximizes total welfare (sum of utilities).
Ayn Rand	Don’t let anyone distract you from your goal (probably D, could be C)
Nietzsche	Don’t fall for “herd morality!” You choose your own morals.

A “Rational” Argument

		Steven	
		Split	Steal
Sarah	Split	\$50k, \$50k	\$0, \$100k
	Steal	\$100k, \$0	\$0, \$0

Sarah’s amount listed first

Sarah can’t control Steven. But she can consider both cases. **Contingent thinking**:

- Contingency #1: Steven picks **Steal**:
- Contingency #2: Steven picks **Split**:
- The strategy that’s best in *every* contingency is: ____________________

What’s Wrong With That Argument?

The argument snuck in an assumption.

What’s Wrong With That Argument?

We assumed Sarah is **selfish and greedy**

- **Selfish:** Her objective only involves her own payoff, not his
- **Greedy:** “More is better.” Her objective is to maximize the thing she cares about

That might be false! Sarah might want to:

Motive	**Selfish?**	**Greedy?**
Maximize sum of payoffs (utilitarian)		
Maximize his payoff (benevolent)		
Minimize the difference (fairness)		
Minimize his payoff (spite)		

Under some of these, the “rational” recommendation flips completely.

Moral: We can’t say what’s “rational” without knowing the person’s objectives

Frames vs. Games

What the table gave us:

- the players ($i \in \{1, 2\}$)
- what each can do ($s_i \in \{C, D\}$)
- outcome function: **outcome** for each strategy profile (e.g., $(D, C) \mapsto$ (\$100k,\$0k))

What the table did *not* give us:

- how each player **ranks** those outcomes
  The player's **objective** or **preferences**

Table alone $=$ **game-frame** (a.k.a. **game form**)

Frame $+$ rankings $=$ **game**

New Example: Meet-Cute Sunday Morning

Alex and Blair meet at a Saturday night party and hit it off. They agree to meet at 11am for a big breakfast, but then both go home and fall asleep without charging their phones. 10:50am: Both wake up, and remember that they talked about (B) Buckeye Donuts, and (H) HangOverEasy. But no way to message and it's time to go! What to do?

- Alex chooses: Buckeye Donuts (B) or HangOverEasy (H).
- Blair chooses: Buckeye Donuts (B) or HangOverEasy (H)
- They choose "simultaneously" (neither sees the other's choice)

Four **outcomes** can happen:

- o_1: both at Buckeye Donuts
- o_2: Alex at BuckyDs, Blair at HangOE (they eat alone)
- o_3: Alex at HangOE, Blair at BuckyDs (they eat alone)
- o_4: both at HangOverEasy

Sunday Morning as a Frame

- Players: $I = \{1, 2\}$ $(1 = \text{Alex}, 2 = \text{Blair})$
- Strategies: $S_1 = \{B, H\}$, $S_2 = \{B, H\}$
- Profiles: $S = S_1 \times S_2 = \{(B,B), (B,H), (H,B), (H,H)\}$
- Outcomes: $O = \{o_1, o_2, o_3, o_4\}$
- Outcome function: $f : S \to O$
  $(B,B) \mapsto o_1$, $(B,H) \mapsto o_2$, $(H,B) \mapsto o_3$, $(H,H) \mapsto o_4$
  Matrix representation:

	B	H
B	o_1	o_2
H	o_3	o_4

But what do they **prefer**? Being together, or going to their favorite place?

Definition: Game-Frame in Strategic Form

A **game-frame in strategic form** is a list of four items

$$\langle I,\ (S_1, S_2, \ldots, S_n),\ O,\ f\rangle$$

where

- $I = \{1, 2, \ldots, n\}$ is the set of **players** ($n \geq 2$)
- S_i is the set of **strategies** available to player i
    - $S = S_1 \times S_2 \times \cdots \times S_n$ is the set of **strategy profiles**
    - a typical profile is $s = (s_1, \ldots, s_n)$
- O is the set of **outcomes**
- $f : S \to O$ is the **outcome function**

Why a function?

Preferences

The Problem

Outcomes are *things*, not numbers:

- “we eat together at Buckeye Donuts”
- “Sarah gets $100,000 and Steven gets nothing”
- “the park gets built and I pay $40”

We can’t add them, average them, or take derivatives.

All we will assume a player can do is **compare** them, two at a time.

One Relation to Rule Them All

For each player i we write down a single relation on O:

$$o \succsim_i o'$$

"Player i considers o to be **at least as good as** o'"
or: "Player i **weakly prefers** o to o'"

This one relation is enough. From it we *define* the other two:

- **Weak preference:** $o \succsim_i o'$: o is *at least as good as* o'
- **Strict preference:** $o \succ_i o'$: o is *strictly better than* o'
- **Indifference:** $o \sim_i o'$: o and o' are *equivalent*; I'm fine with either

You’ve Done This Before

You already know a relation that works exactly this way: $\geq$ on numbers.

Numbers	Outcomes	
$x \geq y$	$o \succsim_i o'$	at least as good as
$x > y$	$o \succ_i o'$	strictly better than
$x = y$	$o \sim_i o'$	just as good as

We can derive $>$ and $=$ from $\geq$:

- $x > y$ means $x \geq y$ and *not* $y \geq x$
- $x = y$ means $x \geq y$ and $y \geq x$

Similarly, we can derive $\succ_i$ and $\sim_i$ from $\succsim_i$:

- $x \succ_i y$ means $x \succsim_i y$ and *not* $y \succsim_i x$
- $x \sim_i y$ means $x \succsim_i y$ and $y \succsim_i x$

Where the Analogy Breaks

With numbers, "at least as big" comes for free:

- any two numbers *can* be compared
- the comparisons *never* cycle (can't have $3 > 2 > 1 > 3$)

With outcomes, neither is automatic. "Both at Buckeye Donuts" and "Sarah gets \$100,000, Steve gets \$0" are not on a number line.

So the two properties we get for free with $\geq$ become **assumptions** about $\succsim_i$.

That's next.

Reading the Symbols

Notation	Interpretation
$o \succsim_i o'$	i finds o at least as good as o' (better than, *or* just as good as)
$o \succ_i o'$	i finds o better than o'; i *strictly prefers* o to o'
$o \sim_i o'$	i finds o just as good as o'; i is *indifferent*

The subscript i matters: preferences belong to **a person**.

Let C be dinner at Cazuela's and Y be dinner at Yoshi Sushi. Then

$$C \succ_{\text{Alice}} Y \quad \text{but} \quad C \sim_{\text{Bob}} Y$$

is perfectly consistent: Alice prefers Cazuela's, Bob truly doesn't care.

Chains are read left to right, best to worst:

$$o_3 \succ o_1 \succ o_2 \sim o_4$$

Two Assumptions We Will Always Make

1. **Completeness.** For any two outcomes o and o':

$$\text{either} \quad o \succsim_i o' \quad \text{or} \quad o' \succsim_i o \quad \text{(or both)}$$

- i.e., the player is always able to make a choice
- "I don't care" $\mapsto$ indifference
- "I don't know" $\mapsto$ violation of completeness

2. **Transitivity.** For any three outcomes:

$$\text{if } o_1 \succsim_i o_2 \; \text{ and } \; o_2 \succsim_i o_3 \; \text{ then } \; o_1 \succsim_i o_3$$

- The ranking doesn't cycle
- Violation: $o_1 \succsim_i o_2 \succsim_i o_3 \succsim_i o_1$

Why Transitivity?

Suppose your preferences *cycle*: $x \succ y$, $y \succ z$, but $z \succ x$.

You own z. I own x and y.

- “Pay me a dollar and I’ll give you y for z”
- “Pay me a dollar and I’ll give you x for y”
- “Pay me a dollar and I’ll give you z for x”

Now you’re back where you started, three dollars poorer. Repeat forever. “Money pump.”

Transitivity Buys Us a Lot

If $\succsim_i$ is complete and transitive, then $\succ_i$ and $\sim_i$ will be transitive as well:

- if $o_1 \sim o_2$ and $o_2 \sim o_3$ then $o_1 \sim o_3$
- if $o_1 \succ o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$
- if $o_1 \succ o_2$ and $o_2 \sim o_3$ then $o_1 \succ o_3$
- if $o_1 \sim o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$

Together these let us line the outcomes up from best to worst, with ties allowed.

Three Ways to Write the Same Ranking

Take $O = \{o_1, o_2, o_3, o_4, o_5\}$ and the ranking

$$o_3 \ \succ \ o_5 \sim o_1 \ \succ \ o_4 \ \succ \ o_2$$

Way 1: chain notation. Exactly what's written above.

Way 2: a vertical list, best on top. Ties share a row.

$$\begin{array}{rl} \text{best} & o_3 \\ & o_1,\ o_5 \\ & o_4 \\ \text{worst} & o_2 \end{array}$$

Three Ways to Write the Same Ranking

Way 3: attach a number to each outcome.

outcome	o_1	o_2	o_3	o_4	o_5
U	6	1	8	2	6

Rule: bigger number $=$ better; equal numbers $=$ indifferent.

Check it against $o_3 \succ o_5 \sim o_1 \succ o_4 \succ o_2$:

(There's a 4th way in the book using sets. Skip it.)

Definition: Ordinal Utility Function

Let $\succsim$ be a complete and transitive ranking of a finite set O.

A function $U : O \to \mathbb{R}$ is an **ordinal utility function representing** $\succsim$ if, for all outcomes o and o':

$$U(o) > U(o') \text{ if and only if } o \succ o'$$

$$U(o) = U(o') \text{ if and only if } o \sim o'$$

Equivalently: $\quad o \succsim o' \text{ if and only if } U(o) \geq U(o').$

$U(o)$ is called the **utility** of outcome o.

Utility Numbers Are Arbitrary

All three of these represent the *same* ranking $o_3 \succ o_1 \succ o_2 \sim o_4$:

	o_1	o_2	o_3	o_4
Utility function 1:	5	2	10	2
Utility function 2:	0.8	0.7	1	0.7
Utility function 3:	27	1	100	1

There are infinitely many. Any order-preserving relabeling works.

This is what "**ordinal**" means: *only the order carries information*.

Consequence: utility cannot be compared across people.

Alice's utility for Massey's pizza is 5. Bob's is 4.

- Does Alice like Massey's more than Bob does? **No idea.**
- Bob could have written 400 instead of 4 and said the same thing
- Each person's numbers only speak about *that person's own* ranking

What Utility Does *Not* Mean

"Alice's utility for Cazuela's is 10."

- Meaningless on its own.

"Alice's utility for Cazuela's is 10 and for Yoshi Sushi is 5."

- Meaningful: it says she prefers Cazuela's.
- Does **not** say Cazuela's is "twice as good."
- We could have used 100 and -25 and said exactly the same thing.

Also: a utility of 1 does not mean "first choice." Low number, low rank.

Think of it as an index of satisfaction, but hold that thought loosely.

Careful: Two Things That Look Alike

Ordinal utility (now)	**Cardinal utility (Ch. 5)**
Only order matters	Differences matter too
Any increasing relabeling is an equally good U	Only positive affine transformations $aU + b$
Enough for dominance, Nash equilibrium	Needed for randomization and expected utility

For the next several weeks, we live entirely in the left column.

Definition: Ordinal Game in Strategic Form

An **ordinal game in strategic form** is

$$\langle I,\ (S_1, \ldots, S_n),\ O,\ f,\ (\succsim_1, \ldots, \succsim_n)\rangle$$

where

- $\langle I, (S_1, \ldots, S_n), O, f\rangle$ is a game-frame, and
- each $\succsim_i$ is a complete and transitive ranking of O

The only new ingredient is the list of rankings, one per player.

Writing Games With Payoffs

	C	D
C	o_1	o_2
D	o_3	o_4

$$U_1(o_1) = 3 \quad U_2(o_1) = 3$$
$$U_1(o_2) = 0 \quad U_2(o_2) = 4$$
$$U_1(o_3) = 4 \quad U_2(o_3) = 0$$
$$U_1(o_4) = 0 \quad U_2(o_4) = 0$$

	C	D
C	3, 3	0, 4
D	4, 0	0, 0

Game Frame $+$ Utilities $=$ "Reduced" Ordinal Game

- Physical outcomes are replaced with their utility values
- We often call these "payoffs" (though they're utilities)
    - Sarah doesn't *really* have a utility for $s = (C, C)$
    - She *actually* has a utility for $o_1 =$ ($50k,$50k)
    - But we can write $U(o_1)$ in the (C, C) cell since $(C, C) \mapsto o_1$
- Convention: Player 1's payoff first, Player 2's payoff second
- Most game theory texts skip the game frame and just write reduced ordinal games!

Payoff Functions

Replace each ranking $\succsim_i$ with a utility function U_i that represents it.

For any strategy profile s, define player i's **payoff function** at s by

$$\pi_i(s) \;=\; U_i\big(f(s)\big)$$

Read "$U_i(f(s))$" from inside to outside:

1. s is a strategy profile (e.g, (C, C))
2. $f(s)$ is the outcome at s (e.g., o_1)
3. $U_i(f(s))$ is just $U_i(o_1)$, player i's utility of that outcome (e.g., 3)

The list $\langle I, (S_1, \ldots, S_n), (\pi_1, \ldots, \pi_n)\rangle$ is the **reduced** ordinal game.
"Reduced" because we've thrown away the names of the outcomes.

The Pipeline

$$
\begin{tikzcd}
\boxed{\text{strategy profile } s} \arrow[r, "f"] \arrow[rr, dashed, bend right, "\pi_i"'] & \boxed{\text{outcome } f(s)} \arrow[r, "U_i"] & \boxed{\text{number } U_i(f(s))}
\end{tikzcd}
$$

Everything from here on is written in terms of π_i

Usually we'll skip the physical outcomes.

But remember that they're there! Under the surface.

Sunday Morning: Three Different Games, One Frame

Story A. Both want to eat together; Alex wants donuts, Blair wants chicken & waffles.

- o_1: both at Buckeye Donuts
- o_2: Alex at BuckyDs, Blair at HangOE (they eat alone)
- o_3: Alex at HangOE, Blair at BuckyDs (they eat alone)
- o_4: both at HangOverEasy

- Alex ($i = 1$): $o_1 \succ_1 o_4 \succ_1 o_2 \succ_1 o_3$
- Blair ($i = 2$): $o_4 \succ_2 o_1 \succ_2 o_3 \succ_2 o_2$

An ordinal game:

Alex \ Blair	BuckyDs	HangOE
BuckyDs	4, 3	2, 1
HangOE	1, 2	3, 4

Sunday Morning: Three Different Games, One Frame

Story B. Alex wants company; Blair wants to eat alone this morning.

- Alex: $o_1 \sim_1 o_4 \succ_1 o_2 \sim_1 o_3$
- Blair: $o_2 \sim_2 o_3 \succ_2 o_1 \sim_2 o_4$

Alex \ Blair	BuckyDs	HangOE
BuckyDs	2, 0	0, 2
HangOE	0, 2	2, 0

Same frame. Completely different game.

Sunday Morning: Three Different Games, One Frame

Story C. Neither cares about company at all; each just wants their favorite food.

- Alex: $o_1 \sim_1 o_2 \succ_1 o_3 \sim_1 o_4$ (donuts, period)
- Blair: $o_2 \sim_2 o_4 \succ_2 o_1 \sim_2 o_3$ (chicken & waffles, period)

		Blair	
		BuckyDs	HangOE
Alex	BuckyDs	1, 0	1, 1
	HangOE	0, 0	0, 1

Here the answer feels obvious. Why?

Back to Golden Balls

Back to Golden Balls. Suppose **both** are selfish and greedy:

- Sarah: $o_3 \succ_1 o_1 \succ_1 o_2 \sim_1 o_4$
- Steven: $o_2 \succ_2 o_1 \succ_2 o_3 \sim_2 o_4$

Pick utilities (let's use "my utility $=$ my dollars" (divided by 1,000)):

Strategy Profile:	(C, C)	(C, D)	(D, C)	(D, D)
Outcome:	$o_1 = (\$50k, \$50k)$	$o_2 = (\$0, \$100k)$	$o_3 = (\$100k, \$0)$	$o_4 = (\$0, \$0)$
U_1	50	0	100	0
U_2	50	100	0	0

Sarah \ Steven	Split	Steal
Split	50, 50	0, 100
Steal	100, 0	0, 0

Back to Golden Balls

Now suppose Sarah is still selfish and greedy...
but Steven is **fair-minded and benevolent** (utility boost $+75$ if same payoffs)

	$o_1 = (\$50k, \$50k)$	$o_2 = (\$0, \$100k)$	$o_3 = (\$100k, \$0)$	$o_4 = (\$0, \$0)$
Sarah's U_1	50	0	100	0
Steven's U_2	$50 + 75 = 125$	100	0	$0 + 75 = 75$

Sarah \ Steven	Split	Steal
Split	50, 125	0, 100
Steal	100, 0	0, 75

Same show, same rules, same table of prizes. **Different game.**
And Steven no longer has an "obviously rational" strategy!

Dominance

Underlining Best Responses

		Steven: Split	Steven: Steal
Sarah	Split	50, 125	0, 100
Sarah	Steal	100, 0	0, 75

How to analyze this?

- Rationality: engage in contingent thinking.
- "What's my best response to every contingency?" (Contingency = strategy of other)
- Look at payoffs in that contingency, underline the highest payoffs
- We call those "best replies" or "best responses"
- If one strategy is *always* a best response then it **dominates** the other
  - **Weakly** dominates: always best response but with some ties
  - **Strictly** dominates: always best response, no ties
  - **Equivalent**: tied in every contingency

Comparing Strategies

		Player 2 L	Player 2 C	Player 2 R
Player 1	A	4, ·	3, ·	5, ·
	B	2, ·	1, ·	4, ·
	C	4, ·	1, ·	5, ·
	D	2, ·	1, ·	4, ·

Look at Player 1 and compare any two strategies (call them x and y)

- If x always beats y w/ no ties: x **strictly dominates** y
- If x always beats y but w/ ties: x **weakly dominates** y
- If they always tie: x and y are **equivalent**

Then do the same for Player 2...

Recommendation/Prediction: Never play a strategy that is dominated!

Notation for "Everyone Else"

With n players, a profile is $s = (s_1, \dots, s_n)$.

Fix a player i. Write

$$s_{-i} = \text{the strategies of all players other than } i$$

so that $s = (s_i,\ s_{-i})$.

- S_{-i} is the set of all such sub-profiles: $S_{-i} = \times_{j \neq i} S_j$
- $\pi_i(s_i, s_{-i})$ means: i plays s_i, everyone else plays s_{-i}

Example: $S_1 = \{a, b, c\}$, $S_2 = \{d, e\}$, $S_3 = \{f, g\}$, and $s = (b, d, g)$.

Then $s_{-2} =$ ____________________
and $(e, s_{-2}) =$ ____________________

Definition: Dominance

9/8

Let i be a player and $a, b \in S_i$ two of their strategies.

- a **strictly dominates** b if

$$\pi_i(a, s_{-i}) > \pi_i(b, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

No ties

- a **weakly dominates** b if

$$\pi_i(a, s_{-i}) \geq \pi_i(b, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

$$\textit{and } \exists\, s_{-i} \text{ s.t. } \pi_i(a, s_{-i}) > \pi_i(b, s_{-i})$$

Some ties

- a is **equivalent** to b if

$$\pi_i(a, s_{-i}) = \pi_i(b, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

all ties

Two Conventions

1. Strict dominance technically satisfies the definition of weak dominance.

→ there is a tie

From here on, "*a* weakly dominates *b*" means *weakly but not strictly*.

2. "Dominated" is a comparison, so name the other strategy.

- "*x* is dominated" $\rightsquigarrow$ dominated by *what*?
- "*x* is dominated by *y*" $\rightsquigarrow$ fine

Think of "dominates" as "is better than in every contingency."

Dominant = Best

"Dominant": *dominates every other strategy.*

- s_i is a **strictly dominant strategy** for i if s_i strictly dominates *every* other strategy of player i
- s_i is a **weakly dominant strategy** if, for every other strategy s_i', either s_i weakly dominates s_i' or s_i is equivalent to s_i'

Consequences:

- There is at most **one** strictly dominant strategy. Unique
- There can be **several** weakly dominant strategies, but any two of them must be equivalent.

Many games: no dominance (RPS)

Only One Can Be Strictly Dominant

Player 1's payoffs only:

		Player 2		
		L	C	R
Player 1	T	4, ·	3, ·	5, ·
	M	2, ·	1, ·	4, ·
	B	1, ·	1, ·	2, ·

T strictly dominates M and B

So T is strictly dominant

unique

T beats M in every column, and beats B in every column. So T is strictly dominant.

Only One Can Be Strictly Dominant

It cannot be done, and the reason is short.

Suppose a and b are both strictly dominant, and $a \neq b$.

- a strictly dominant $\Rightarrow$ $\pi_i(a, s_{-i}) > \pi_i(b, s_{-i})$ everywhere
- b strictly dominant $\Rightarrow$ $\pi_i(b, s_{-i}) > \pi_i(a, s_{-i})$ everywhere

Contradiction!!

contradiction. only 1 strictly dominant

(Technically, would be a transitivity violation: $f(a, s_{-i}) \succ_i f(b, s_{-i}) \succ_i f(a, s_{-i})$)

Several Can Be Weakly Dominant

Player 1 \ Player 2	L	R
T	3, ·	2, ·
M	3, ·	2, ·
B	1, ·	2, ·

equivalent

always BR

always BR

ties ⇒ weak dominance

- T vs B: $3 > 1$ and $2 = 2$, so T **weakly dominates** B
- M vs B: same story
- T vs M: identical in both columns, so they are **equivalent**

Check the definition: for every other strategy, T either weakly dominates it or is equivalent to it. Same for M.

So T and M are *both* weakly dominant.

But They Must Be Equivalent

Notice that T and M in that game were the same row twice. That is not a coincidence.

Suppose a and b are both weakly dominant.

- Then a weakly dominates b, so $\pi_i(a, s_{-i}) \geq \pi_i(b, s_{-i})$ for *every* s_{-i}
- And b weakly dominates a, so $\pi_i(b, s_{-i}) \geq \pi_i(a, s_{-i})$ for *every* s_{-i}

So the only way to have two weakly dominant strategies is for them to be equivalent:

$$\underbrace{\pi_i(a, s_{-i}) = \pi_i(b, s_{-i})}_{\text{equiv.}} \quad \forall s_{-i}$$

Dominant-Strategy Profiles

Let $s = (s_1, \ldots, s_n)$ be a strategy profile.

- s is a **strict dominant-strategy profile** if s_i is strictly dominant for *every* player i
- s is a **weak dominant-strategy profile** if s_i is weakly dominant for every player i, and for at least one player j, s_j is not *strictly* dominant

Golden balls: (D,D) is a weak dom. strat. profile

Unqualified "dominant-strategy profile" defaults to **weak**.

(You'll also see "dominant-strategy equilibrium" in the literature.)

Your Turn

Only Player 1's payoffs are shown.

		Player 2		
		E	F	G
Player 1	A	3, ·	2, ·	1, ·
	B	2, ·	1, ·	0, ·
	C	3, ·	2, ·	1, ·
	D	2, ·	0, ·	0, ·

weakly dominant
$\Downarrow$
equivalent

Classify each pair. Then: any weakly dominant strategies?

A vs. *B*:

A vs. *D*:

A vs. *C*:

B vs. *D*:

So What About Sarah?

Selfish and greedy version:

		Steven: Split	Steven: Steal
Sarah	Split	50, 50	0, 100
Sarah	Steal	100, 0	0, 0

tie

← always BR

weak dom strat

- Steal is a weak dominant strategy for each player
- So (Steal, Steal) is a weakly dominant-strategy profile

What if both are fair-minded (+75 for equal earnings)?

		Steven: Split	Steven: Steal
Sarah	Split	125, 125	0, 100
Sarah	Steal	100, 0	75, 75

no dominant strategy

Our Own Example: The Price War

Dos Hermanos and Pitabilities park on the same block every day. Each morning, each owner independently sets a **Normal** price or runs a **Discount**.

- If neither discounts: they split the lunch crowd at healthy margins
- If exactly one discounts: that truck pulls most of the crowd
- If both discount: they split the same crowd at thin margins

outcomes

game frame

Each owner ranks the four outcomes:

utility:

$$\underset{3}{\text{(I discount, they don't)}} \succ \underset{2}{\text{(neither)}} \succ \underset{1}{\text{(both)}} \succ \underset{0}{\text{(they discount, I don't)}}$$

prefs

game frame + prefs = game

The Price War

		Pitabilities: Normal	Pitabilities: Discount
Dos Hermanos	Normal	2, 2	0, 3
	Discount	3, 0	1, 1

no Pareto superiority

← always BR
strictly dominant strategy

- Discount ____strictly____ dominates Normal, for both
- So (Discount, Discount) is a ____strictly dominant strategy____ profile

But look at (Normal, Normal): higher profits

normal 2
discount 1

Solution Concepts

My one beef with Bonanno:

- Given a game, what should we prescribe rational players to do?
    1. Only play a dominant strategy
    2. Don't play dominated strategies undominated strategies
    3. Play Nash equilibrium
    4. Only play subgame perfect Nash equilibria
    5. ...
- These are all called **solution concepts**

- Suppose we use dominant strategies
    - Rock Paper Scissors has no dominant strategy!
    - What do we think people *should* do when there's no dominant strategy?

Evaluating Outcomes: Pareto superiority

Definition: Pareto Superiority

Let o and o' be two outcomes.

unambiguously strictly better for everyone

- o is **strictly Pareto superior** to o' if *every* player prefers o to o': $o \succ_i o'$ for all i
- o is **weakly Pareto superior** to o' if every player finds o at least as good, and at least one strictly prefers it

with ties for some player i

$o \sim_i o'$

$o \succsim_i o'$

In reduced games, same thing with payoffs: $\pi_i(s) > \pi_i(s')$ for all i, etc.

In the Price War, (Normal, Normal) is strictly Pareto superior to (Discount, Discount).

Yet rational play lands on (Discount, Discount).

The Prisoner's Dilemma

That structure has a name. Bonanno's version:

Doug and Ed are the only two candidates for a best-worker prize, currently tied. Unless one outperforms the other, nobody gets it. Each chooses Normal or Extra effort.

- o_1: nobody gets the prize, nobody sacrifices family time
- o_2: Ed gets the prize and sacrifices family time, Doug does not
- o_3: Doug gets the prize and sacrifices family time, Ed does not
- o_4: nobody gets the prize, both sacrifice family time

Both are willing to sacrifice family time for the prize, otherwise value it, and are envious: $o_3 \succ_{\text{Doug}} o_1 \succ_{\text{Doug}} o_4 \succ_{\text{Doug}} o_2$.

The Prisoner's Dilemma

Doug \ Ed	Normal	Extra
Normal	2, 2	0, $\underline{3}$
Extra	$\underline{3}$, 0	$\underline{1}$, $\underline{1}$

Extra effort is strictly dominant for both.
(Extra, Extra) is a strict dominant-strategy profile.

Same game as the price war: different story, identical structure.
That is what game theory is for.

Individual vs. Collective Rationality

A strictly dominant strategy is not a suggestion. Playing anything else means a lower payoff *no matter what the other player does*.

So rational individual play $\Rightarrow$ (Extra, Extra), even though both prefer (Normal, Normal).

Couldn't they just agree?

No coordination

- Non-cooperative game theory assumes agreements are **not binding**
- If I expect you to keep it, I gain by breaking it

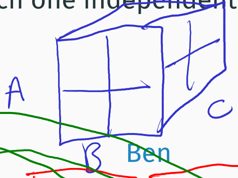
- If I expect you to break it, I gain by breaking it too

Let’s Work One Out

Three Booths at the North Market

Ann, Ben, and Cal each run a booth. Every Saturday each one independently picks a special: the **Classic** or something **New**.
Three player game: third player picks the **matrix**.

2x2x2 game



Cal: Classic

Ann \ Ben	Classic	New
Classic	3, 3, 2	4, 3, 1
New	1, 3, 4	2, 1, 3

the

Cal: New

Ann \ Ben	Classic	New
Classic	3, 2, 3	4, 0, 2
New	1, 2, 3	2, 0, 4

Strict D.S.

Ben: classic is weak D.S.

Ann picks the row, Ben the column, Cal the table. Payoffs in order: Ann, Ben, Cal.

Cal: No dom. strategy

There is NO dom. strategy profile

Three Booths at the North Market

1. Does Ann have a dominant strategy? Strict or weak?
2. Does Ben?
3. Does Cal?
4. Is there a dominant-strategy profile?

   No

5. If Ann and Ben play their dominant strategies, what should Cal do?

What Have We Learned, Charlie Brown?

- A **game-frame** is players, strategies, outcomes, and f
- Preferences are a complete, transitive relation $\succsim_i$ on outcomes
- An **ordinal utility function** represents $\succsim_i$; the numbers are arbitrary
- Frame $+$ preferences $=$ **game**; composing gives payoffs $\pi_i = U_i \circ f$
- **Dominance** compares two strategies of one player across all contingencies
- Our first solution concept: dominant strategies

Next: what if nobody has a dominant strategy?

Reading & Practice

- Bonanno, *Game Theory*, §2.1–2.2
- Exercises: §2.9.1 (frames and preferences), §2.9.2 (dominance)
- Full solutions are in §2.10. Try them before you look