

Chapter 2 (Part 1): Ordinal Games in Strategic Form

ECON 5001

Prof. Paul J. Healy

Based on: *Game Theory* by Giacomo Bonanno

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Click me: [The Golden Balls Video](https://www.facebook.com/SebTalksBoxing/videos/woman-wins-100k-with-sensational-shthousery/1127818344236313/)

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Golden Balls

A British game show. Two contestants, Sarah and Steven, have built up a jackpot of \$100,000.

Each secretly picks one of two balls:

- one says **Split**
- one says **Steal**

They choose **simultaneously**. Then both are revealed.

- Both Split: each gets \$50,000
- One Steals: the stealer gets all \$100,000, the other gets nothing
- Both Steal: nobody gets anything

Golden Balls: The Table

		Steven	
		Split	Steal
Sarah	Split	Sarah: \$50k Steven: \$50k	Sarah: \$0 Steven: \$100k
	Steal	Sarah: \$100k Steven: \$0	Sarah: \$0 Steven: \$0

Each row is a choice for Sarah, each column a choice for Steven.
Each cell is what **happens** (the “outcome”)

The most studied game in game theory:

Prisoner's Dilemma

Two players, each with a **cooperate** option and a **defect** option, where

- defecting looks better *no matter what* the other player does, and
- if both defect, both end up worse off than if both had cooperated

Split = cooperate. Steal = defect.

The Same Dilemma, Out in the Wild

Messier stories, same skeleton (**cooperate** vs. **defect**):

- **Arms races:** hold your build-up vs. out-arm them — both spend fortunes, neither safer
- **Carbon emissions:** cut vs. free-ride on everyone else's cuts
- **A shared fishery:** fish your quota vs. overfish — the stock collapses for all
- **Doping in sport:** race clean vs. dope — if everyone dopes, the medals go to the same people, at a cost
- **Price wars:** hold your price vs. undercut the rival — margins vanish on both sides
- **Ad spending:** modest campaign vs. outspend the rival — market shares barely move
- **Antibiotics:** prescribe sparingly vs. prescribe on demand — resistance is everyone's problem
- **Attack ads:** stay positive vs. go negative — both get dragged down
- **Standing at a concert:** stay seated vs. stand for a better view — everyone stands, nobody sees more

Theory vs. Reality

The gaps between theory and reality:

- **many** players, not two (fishing fleets, countries, fans in a stadium)
- **degrees** of cooperation, not two buttons (how much to emit, how far to cut price)
- **repetition**, reputation, and the chance to retaliate later
- **imperfect monitoring**: you can't always tell who defected
- **outside enforcement**: treaties, regulators, drug testing, cartel policing

None of that changes the core tension:

what's best for me, taken alone, leaves us all worse off.

Before We Analyze It: Let's Play

- Phone, tablet, or laptop (whatever you've got)
- Log in to **MobLab** and join our session
- Class code: `dv1cf8ft`
- One decision: **Split** or **Steal** (Cooperate or Defect)

No talking, no signaling. Choose what you'd actually choose.

For Fun: What Would the Philosophers Say? (Not on Quiz/Test)

Philosopher	My (very amateurish) Understanding
Kant	Categorical imperative: What if <i>everyone</i> did that? Thus, $C \succ D$
Hobbes	Life is “solitary, poor, nasty, brutish, and short.” We need a government
Locke	People have a natural duty to each other, even without government
Hume	Cooperate becomes convention through repeated play (ex: two farmers)
Rousseau	Trust and the social contract keep society functioning. Play C
Bentham/Mill	Cooperate because it maximizes total welfare (sum of utilities).
Ayn Rand	Don’t let anyone distract you from your goal (probably D , could be C)
Nietzsche	Don’t fall for “herd morality!” You choose your own morals.

A “Rational” Argument

		Steven	
		Split	Steal
Sarah	Split	\$50k, \$50k	\$0, \$100k
	Steal	\$100k, \$0	\$0, \$0

Sarah's amount listed first

Sarah can't control Steven. But she can consider both cases. **Contingent thinking:**

- Contingency #1: Steven picks **Steal**:
- Contingency #2: Steven picks **Split**:
- The strategy that's best in every contingency is: _____

What's Wrong With That Argument?

The argument snuck in an assumption.

What's Wrong With That Argument?

We assumed Sarah is **selfish and greedy**

- **Selfish:** Her objective only involves her own payoff, not his
- **Greedy:** “More is better.” Her objective is to maximize the thing she cares about

That might be false! Sarah might want to:

Motive	Selfish?	Greedy?
Maximize sum of payoffs (utilitarian)		
Maximize his payoff (benevolent)		
Minimize the difference (fairness)		
Minimize his payoff (spite)		

Under some of these, the “rational” recommendation flips completely.

Moral: We can't say what's “rational” without knowing the person's objectives

Frames vs. Games

What the table gave us:

- the players ($i \in \{1, 2\}$)
- what each can do ($s_i \in \{C, D\}$)
- outcome function: **outcome** for each strategy profile (e.g., $(D, C) \mapsto (\$100k, \$0k)$)

What the table did *not* give us:

- how each player **ranks** those outcomes

The player's **objective** or **preferences**

Table alone = **game-frame** (a.k.a. **game form**)

Frame + rankings = **game**

New Example: Meet-Cute Sunday Morning

Alex and Blair meet at a Saturday night party and hit it off. They agree to meet at 11am for a big breakfast, but then both go home and fall asleep without charging their phones. 10:50am: Both wake up, and remember that they talked about (B) Buckeye Donuts, and (H) HangOverEasy. But no way to message and it's time to go! What to do?

- Alex chooses: Buckeye Donuts (*B*) or HangOverEasy (*H*).
- Blair chooses: Buckeye Donuts (*B*) or HangOverEasy (*H*)
- They choose “simultaneously” (neither sees the other's choice)

Four **outcomes** can happen:

- o_1 : both at Buckeye Donuts
- o_2 : Alex at BuckyDs, Blair at HangOE (they eat alone)
- o_3 : Alex at HangOE, Blair at BuckyDs (they eat alone)
- o_4 : both at HangOverEasy

Sunday Morning as a Frame

- Players: $I = \{1, 2\}$ (1 = Alex, 2 = Blair)
- Strategies: $S_1 = \{B, H\}$, $S_2 = \{B, H\}$
- Profiles: $S = S_1 \times S_2 = \{(B, B), (B, H), (H, B), (H, H)\}$
- Outcomes: $O = \{o_1, o_2, o_3, o_4\}$
- Outcome function: $f : S \rightarrow O$
 $(B, B) \mapsto o_1, (B, H) \mapsto o_2, (H, B) \mapsto o_3, (H, H) \mapsto o_4$

Matrix representation:

	B	H
B	o_1	o_2
H	o_3	o_4

But what do they **prefer**? Being together, or going to their favorite place?

Definition: Game-Frame in Strategic Form

A **game-frame in strategic form** is a list of four items

$$\langle I, (S_1, S_2, \dots, S_n), O, f \rangle$$

where

- $I = \{1, 2, \dots, n\}$ is the set of **players** ($n \geq 2$)
- S_i is the set of **strategies** available to player i
 - $S = S_1 \times S_2 \times \dots \times S_n$ is the set of **strategy profiles**
 - a typical profile is $s = (s_1, \dots, s_n)$
- O is the set of **outcomes**
- $f : S \rightarrow O$ is the **outcome function**

Why a function?

Preferences

The Problem

Outcomes are *things*, not numbers:

- “we eat together at Buckeye Donuts”
- “Sarah gets \$100,000 and Steven gets nothing”
- “the park gets built and I pay \$40”

We can't add them, average them, or take derivatives.

All we will assume a player can do is **compare** them, two at a time.

One Relation to Rule Them All

For each player i we write down a single relation on O :

$$o \succsim_i o'$$

“Player i considers o to be **at least as good as** o' ”
or: “Player i **weakly prefers** o to o' ”

This one relation is enough. From it we *define* the other two:

- **Weak preference:** $o \succsim_i o' :$ o is at least as good as o'
- **Strict preference:** $o \succ_i o' :$ o is strictly better than o'
- **Indifference:** $o \sim_i o' :$ o and o' are equivalent; I'm fine with either

You've Done This Before

You already know a relation that works exactly this way: $\geq$ on numbers.

Numbers	Outcomes	
$x \geq y$	$o \succsim_i o'$	at least as good as
$x > y$	$o \succ_i o'$	strictly better than
$x = y$	$o \sim_i o'$	just as good as

We can derive $>$ and $=$ from $\geq$:

- $x > y$ means $x \geq y$ and *not* $y \geq x$
- $x = y$ means $x \geq y$ and $y \geq x$

Similarly, we can derive $\succ_i$ and $\sim_i$ from $\succsim_i$:

- $x \succ_i y$ means $x \succsim_i y$ and *not* $y \succsim_i x$
- $x \sim_i y$ means $x \succsim_i y$ and $y \succsim_i x$

Where the Analogy Breaks

With numbers, “at least as big” comes for free:

- any two numbers *can* be compared
- the comparisons *never* cycle (can’t have $3 > 2 > 1 > 3$)

With outcomes, neither is automatic. “Both at Buckeye Donuts” and “Sarah gets \$100,000, Steve gets \$0” are not on a number line.

So the two properties we get for free with $\geq$ become **assumptions** about $\succsim_i$.

That’s next.

Reading the Symbols

Notation	Interpretation
$o \succsim_i o'$	i finds o at least as good as o' (better than, or just as good as)
$o \succ_i o'$	i finds o better than o' ; i strictly prefers o to o'
$o \sim_i o'$	i finds o just as good as o' ; i is indifferent

The subscript i matters: preferences belong to **a person**.

Let C be dinner at Cazuela's and Y be dinner at Yoshi Sushi. Then

$$C \succ_{\text{Alice}} Y \quad \text{but} \quad C \sim_{\text{Bob}} Y$$

is perfectly consistent: Alice prefers Cazuela's, Bob truly doesn't care.

Chains are read left to right, best to worst:

$$o_3 \succ o_1 \succ o_2 \sim o_4$$

Two Assumptions We Will Always Make

1. Completeness. For any two outcomes o and o' :

either $o \succsim_i o'$ or $o' \succsim_i o$ (or both)

- i.e., the player is always able to make a choice
- “I don’t care” $\mapsto$ indifference
- “I don’t know” $\mapsto$ violation of completeness

2. Transitivity. For any three outcomes:

if $o_1 \succsim_i o_2$ and $o_2 \succsim_i o_3$ then $o_1 \succsim_i o_3$

- The ranking doesn’t cycle
- Violation: $o_1 \succsim_i o_2 \succsim_i o_3 \succsim_i o_1$

Why Transitivity?

Suppose your preferences cycle: $x \succ y$, $y \succ z$, but $z \succ x$.

You own z . I own x and y .

- “Pay me a dollar and I’ll give you y for z ”
- “Pay me a dollar and I’ll give you x for y ”
- “Pay me a dollar and I’ll give you z for x ”

Now you’re back where you started, three dollars poorer. Repeat forever. “Money pump.”

Transitivity Buys Us a Lot

If $\succsim_i$ is complete and transitive, then $\succ_i$ and $\sim_i$ will be transitive as well:

- if $o_1 \sim o_2$ and $o_2 \sim o_3$ then $o_1 \sim o_3$
- if $o_1 \succ o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$
- if $o_1 \succ o_2$ and $o_2 \sim o_3$ then $o_1 \succ o_3$
- if $o_1 \sim o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$

Together these let us line the outcomes up from best to worst, with ties allowed.

Three Ways to Write the Same Ranking

Take $O = \{o_1, o_2, o_3, o_4, o_5\}$ and the ranking

$$o_3 \succ o_5 \sim o_1 \succ o_4 \succ o_2$$

Way 1: chain notation. Exactly what's written above.

Way 2: a vertical list, best on top. Ties share a row.

best	o_3
	o_1, o_5
	o_4
worst	o_2

Three Ways to Write the Same Ranking

Way 3: attach a number to each outcome.

outcome	o_1	o_2	o_3	o_4	o_5
U	6	1	8	2	6

Rule: bigger number = better; equal numbers = indifferent.

Check it against $o_3 \succ o_5 \sim o_1 \succ o_4 \succ o_2$:

(There's a 4th way in the book using sets. Skip it.)

Definition: Ordinal Utility Function

Let $\succsim$ be a complete and transitive ranking of a finite set O .

A function $U : O \rightarrow \mathbb{R}$ is an **ordinal utility function representing** $\succsim$ if, for all outcomes o and o' :

$$U(o) > U(o') \text{ if and only if } o \succ o'$$

$$U(o) = U(o') \text{ if and only if } o \sim o'$$

Equivalently: $o \succsim o'$ if and only if $U(o) \geq U(o')$.

$U(o)$ is called the **utility** of outcome o .

Utility Numbers Are Arbitrary

All three of these represent the *same* ranking $o_3 \succ o_1 \succ o_2 \sim o_4$:

	o_1	o_2	o_3	o_4
Utility function 1:	5	2	10	2
Utility function 2:	0.8	0.7	1	0.7
Utility function 3:	27	1	100	1

There are infinitely many. Any order-preserving relabeling works.

This is what “**ordinal**” means: *only the order carries information*.

Consequence: utility cannot be compared across people.

Alice’s utility for Massey’s pizza is 5. Bob’s is 4.

- Does Alice like Massey’s more than Bob does? **No idea.**
- Bob could have written 400 instead of 4 and said the same thing
- Each person’s numbers only speak about *that person’s own* ranking

What Utility Does *Not* Mean

“Alice’s utility for Cazuela’s is 10.”

- Meaningless on its own.

“Alice’s utility for Cazuela’s is 10 and for Yoshi Sushi is 5.”

- Meaningful: it says she prefers Cazuela’s.
- Does **not** say Cazuela’s is “twice as good.”
- We could have used 100 and -25 and said exactly the same thing.

Also: a utility of 1 does not mean “first choice.” Low number, low rank.

Think of it as an index of satisfaction, but hold that thought loosely.

Careful: Two Things That Look Alike

Ordinal utility (now)	Cardinal utility (Ch. 5)
Only order matters	Differences matter too
Any increasing relabeling is an equally good U	Only positive affine transformations $aU + b$
Enough for dominance, Nash equilibrium	Needed for randomization and expected utility

For the next several weeks, we live entirely in the left column.

Definition: Ordinal Game in Strategic Form

An **ordinal game in strategic form** is

$$\langle I, (S_1, \dots, S_n), O, f, (\succsim_1, \dots, \succsim_n) \rangle$$

where

- $\langle I, (S_1, \dots, S_n), O, f \rangle$ is a game-frame, and
- each $\succsim_i$ is a complete and transitive ranking of O

The only new ingredient is the list of rankings, one per player.

Writing Games With Payoffs

	C	D
C	o_1	o_2
D	o_3	o_4

outcomes
↑

Game Frame

Player 1

$$U_1(o_1) = 3$$

$$U_1(o_2) = 0$$

$$U_1(o_3) = 4$$

$$U_1(o_4) = 0$$

Player 2

$$U_2(o_1) = 3$$

$$U_2(o_2) = 4$$

$$U_2(o_3) = 0$$

$$U_2(o_4) = 0$$

$$o_3 \succ o_1 \succ o_2 \sim o_4$$

$$o_2 \succ o_1 \succ o_3 \sim o_4$$

+ Utilities

9/5

$$U_1(c, c) = 3$$

	C	D
C	3, 3	0, 8
D	4, 0	0, 0

↑

= "Reduced" Ordinal Game

- Physical outcomes are replaced with their utility values

- We often call these "payoffs" (though they're utilities)

- Sarah doesn't really have a utility for $s = (C, C)$

- She actually has a utility for $o_1 = (\$50k, \$50k)$

- But we can write $U(o_1)$ in the (C, C) cell since $(C, C) \mapsto o_1$

- Convention: Player 1's payoff first, Player 2's payoff second

- Most game theory texts skip the game frame and just write reduced ordinal games!

$$s = (c, c)$$

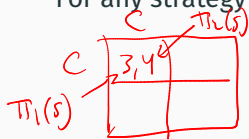
$$\text{outcome function } f(s) = (\$50k, \$50k)$$

$$U_1(f(s)) = U_1(\$50k, \$50k) = 3$$

Payoff Functions

Replace each ranking $\succsim_i$ with a utility function U_i that represents it.

For any strategy profile s , define player i 's **payoff function** at s by



$$\pi_i(s) = U_i(f(s))$$

$\uparrow$ "payoff" at s $\uparrow$ $o_1 = f(c, c)$

$$s = (c, c)$$

$$s_1 = c$$

$$s_2 = c$$

Read " $U_i(f(s))$ " from inside to outside:

1. s is a strategy profile (e.g. (C, C))
2. $f(s)$ is the outcome at s (e.g., o_1)
3. $U_i(f(s))$ is just $U_i(o_1)$, player i 's utility of that outcome (e.g., 3)

Players, Strategies, "Payoffs" (utility)

The list $\langle I, (S_1, \dots, S_n), (\pi_1, \dots, \pi_n) \rangle$ is the **reduced** ordinal game.

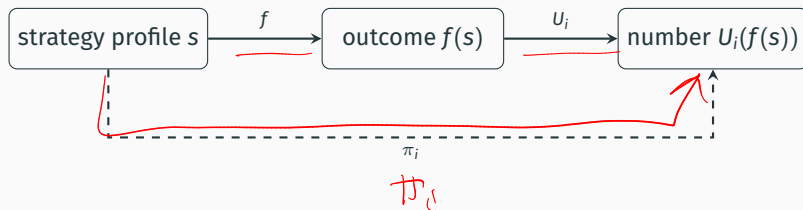
"Reduced" because we've thrown away the names of the outcomes.

$$U_i(o_1) \text{ or } U_i(f(c, c)) \text{ or } U_i(f(s))$$

$$s = (c, c) \quad \pi_1(s) = 3$$

$$s' = (c, d) \quad \pi_1(s') = 2$$

The Pipeline



Everything from here on is written in terms of π_i

Usually we'll skip the physical outcomes.

But remember that they're there! Under the surface.

Sunday Morning: Three Different Games, One Frame

Story A. Both want to eat together; Alex wants ~~donuts~~ ^{HoE}, Blair wants ~~chicken & waffles~~ ^{Buckeye Donuts}.

- o_1 : both at Buckeye Donuts
- o_2 : Alex at BuckyDs, Blair at HangOE (they eat alone)
- o_3 : Alex at HangOE, Blair at BuckyDs (they eat alone)
- o_4 : both at HangOverEasy
- Alex ($i = 1$): $o_1 \succ_1 o_4 \succ_1 o_2 \succ_1 o_3$
- Blair ($i = 2$): $o_4 \succ_2 o_1 \succ_2 o_3 \succ_2 o_2$

An ordinal game:

		Blair	
		BuckyDs	HangOE
Alex	BuckyDs	o_1 4, 3	o_2 2, 1
	HangOE	o_3 1, 2	o_4 3, 4

Sunday Morning: Three Different Games, One Frame

Story B. Alex wants company; Blair wants to eat alone this morning.

- Alex: $\underbrace{o_1 \sim_1 o_4}_{\downarrow} \succ_1 \underbrace{o_2 \sim_1 o_3}$
- Blair: $\underbrace{o_2 \sim_2 o_3} \succ_2 \underbrace{o_1 \sim_2 o_4}$

		Blair	
		BuckyDs	HangOE
Alex	BuckyDs	$\underbrace{o_1}_{\text{red}} \text{ 2, 0}$	$\underbrace{o_2}_{\text{red}} \text{ 0, 2}$
	HangOE	$\underbrace{o_3}_{\text{red}} \text{ 0, 2}$	$\underbrace{o_4}_{\text{red}} \text{ 2, 0}$

Same frame. Completely different game.

Sunday Morning: Three Different Games, One Frame

Story C. Neither cares about company at all; each just wants their favorite food.

- Alex: $o_1 \sim_1 o_2 \succ_1 o_3 \sim_1 o_4$ (donuts, period)
- Blair: $o_2 \sim_2 o_4 \succ_2 o_1 \sim_2 o_3$ (chicken & waffles, period)

		Blair	
		BuckyDs	HangOE
Alex	BuckyDs	<u>1</u> , 0	<u>1</u> , 1
	HangOE	<u>0</u> , 0	<u>0</u> , 1

Here the answer feels obvious. Why?

Back to Golden Balls

Back to Golden Balls. Suppose both are selfish and greedy:

- Sarah: $o_3 \succ_1 o_1 \succ_1 o_2 \sim_1 o_4$
- Steven: $o_2 \succ_2 o_1 \succ_2 o_3 \sim_2 o_4$

Pick utilities (let's use "my utility = my dollars" (divided by 1,000)):

Strategy Profile:	(C, C)	(C, D)	(D, C)	(D, D)
Outcome:	$o_1 = (\$50k, \$50k)$	$o_2 = (\$0, \$100k)$	$o_3 = (\$100k, \$0)$	$o_4 = (\$0, \$0)$
U_1	50	0	100	0
U_2	50	100	0	0

		Steven	
		Split	Steal
Sarah	Split	50, 50	0, 100
	Steal	100, 0	0, 0

Back to Golden Balls

Now suppose Sarah is still selfish and greedy...

but Steven is **fair-minded and benevolent** (utility boost +75 if same payoffs)

	$o_1 = (\$50k, \$50k)$	$o_2 = (\$0, \$100k)$	$o_3 = (\$100k, \$0)$	$o_4 = (\$0, \$0)$
Sarah's U_1	50	0	100	0
Steven's U_2	$50 + 75 = 125$	100	0	$0 + 75 = 75$

↑
boost

↑
boost

Steven Not obvious!

	Split	Steal
Sarah	Split	50, 125
	Steal	100, 0
		0, 100
		0, 75

Assuming
prefs are
"common knowledge"

Same show, same rules, same table of prizes. **Different game.**
And Steven no longer has an "obviously rational" strategy!

Dominance

Underlining Best Responses

Contingency 1
Contingency 2

Sarah

	Split	Steal
Split	50, <u>125</u>	<u>0</u> , 100
Steal	<u>100</u> , 0	<u>0</u> , <u>75</u>

Steven

no strategy dominates the other
weak dominance
BR to both split and steal
both

How to analyze this?

- Rationality: engage in contingent thinking.
- "What's my best response to every contingency?" (Contingency = strategy of other)
- Look at payoffs in that contingency, underline the highest payoffs
- We call those "best replies" or "best responses"
- If one strategy is *always* a best response then it **dominates** the other
 - **Weakly** dominates: always best response but with some ties
 - **Strictly** dominates: always best response, no ties
 - **Equivalent**: tied in every contingency

50	0
100	10

← strictly

0	10
0	10

Comparing Strategies

		Player 2		
		L	C	R
Player 1	A	4, 1	3, 1	5, 1
	B	2, 1	1, 1	4, 1
	C	4, 1	1, 1	5, 1
	D	2, 1	1, 1	4, 1

strictly dom by A
weakly dom by A
strictly dom by A

Player 1

A vs. B
A strictly dom. B ✓

C vs. D
C weakly dom. D

B vs. D
B and D are equivalent

A vs. C
A weakly dom C

Look at Player 1 and compare any two strategies (call them x and y)

- If x always beats y w/ no ties:
- If x always beats y but w/ ties:
- If they always tie:

x **strictly dominates** y
x **weakly dominates** y
x and y are **equivalent**

Then do the same for Player 2...

Recommendation/Prediction: Never play a strategy that is dominated!

Notation for "Everyone Else"

With n players, a profile is $s = (s_1, \dots, s_n)$.

Fix a player i . Write

s_{-i} = the strategies of all players other than i

possible
contingencies

so that $s = (s_i, s_{-i})$.

← everyone else's choice

↑
i's choice

- S_{-i} is the set of all such sub-profiles: $S_{-i} = \times_{j \neq i} S_j$
- $\pi_i(s_i, s_{-i})$ means: i plays s_i , everyone else plays s_{-i}

Example: $S_1 = \{a, b, c\}$, $S_2 = \{d, e\}$, $S_3 = \{f, g\}$, and $s = (b, d, g)$.

Then $s_{-2} = (b, g)$

and $(e, s_{-2}) = (e, (b, g))$

↑
 $s_2 = e$ not d

Definition: Dominance

Let i be a player and a, b $\in S_i$ two of their strategies.

- a strictly dominates b if

$$\pi_i(a, s_{-i}) > \pi_i(b, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

strictly (with arrow pointing to $>$) *"for all"* (with arrow pointing to $\forall$)

NO
TIES

- a weakly dominates b if

$$\pi_i(a, s_{-i}) \geq \pi_i(b, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

weak (possible tie) (with arrow pointing to $\geq$)

$$\text{and } \exists s_{-i} \text{ s.t. } \pi_i(a, s_{-i}) > \pi_i(b, s_{-i})$$

"exists" (with arrow pointing to $\exists$)

$\leftarrow$ not a tie

SOME
TIES
BUT
NOT
ALL

- a is **equivalent** to b if

$$\pi_i(a, s_{-i}) = \pi_i(b, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

always a tie

ALL
TIES

Two Conventions

1. Strict dominance technically satisfies the definition of weak dominance.

From here on, “ a weakly dominates b ” means *weakly but not strictly*.

2. “Dominated” is a comparison, so name the other strategy.

- “ x is dominated” $\rightsquigarrow$ dominated by *what*?
- “ x is dominated by y ” $\rightsquigarrow$ fine

Think of “dominates” as “is better than in every contingency.”

“Dominant”: *dominates every other strategy.*

- s_i is a **strictly dominant strategy** for i if s_i strictly dominates every other strategy of player i
- s_i is a **weakly dominant strategy** if, for every other strategy s'_i , either s_i weakly dominates s'_i or s_i is equivalent to s'_i

Consequences:

- There is at most **one** strictly dominant strategy.
- There can be **several** weakly dominant strategies, but any two of them must be equivalent.

Only One Can Be Strictly Dominant

Player 1's payoffs only:

		Player 2		
		L	C	R
Player 1	T	4, ·	3, ·	5, ·
	M	2, ·	1, ·	4, ·
	B	1, ·	1, ·	2, ·

T beats M in every column, and beats B in every column. So T is strictly dominant.

Only One Can Be Strictly Dominant

It cannot be done, and the reason is short.

Suppose a and b are both strictly dominant, and $a \neq b$.

- a strictly dominant $\Rightarrow \pi_i(a, s_{-i}) > \pi_i(b, s_{-i})$ everywhere
- b strictly dominant $\Rightarrow \pi_i(b, s_{-i}) > \pi_i(a, s_{-i})$ everywhere

Contradiction!!

(Technically, would be a transitivity violation: $f(a, s_{-i}) \succ_i f(b, s_{-i}) \succ_i f(a, s_{-i})$)

Several Can Be Weakly Dominant

		Player 2	
		L	R
Player 1	T	3, ·	2, ·
	M	3, ·	2, ·
	B	1, ·	2, ·

- T vs B : $3 > 1$ and $2 = 2$, so T **weakly dominates** B
- M vs B : same story
- T vs M : identical in both columns, so they are **equivalent**

Check the definition: for every other strategy, T either weakly dominates it or is equivalent to it. Same for M .

So T and M are *both* weakly dominant.

But They Must Be Equivalent

Notice that T and M in that game were the same row twice. That is not a coincidence.

Suppose a and b are both weakly dominant.

- Then a weakly dominates b , so $\pi_i(a, s_{-i}) \geq \pi_i(b, s_{-i})$ for every s_{-i}
- And b weakly dominates a , so $\pi_i(b, s_{-i}) \geq \pi_i(a, s_{-i})$ for every s_{-i}

So the only way to have two weakly dominant strategies is for them to be equivalent:

$$\pi_i(a, s_{-i}) = \pi_i(b, s_{-i}) \quad \forall s_{-i}$$

Dominant-Strategy Profiles

Let $s = (s_1, \dots, s_n)$ be a strategy profile.

- s is a **strict dominant-strategy profile** if s_i is strictly dominant for every player i
- s is a **weak dominant-strategy profile** if s_i is weakly dominant for every player i , and for at least one player j , s_j is not *strictly* dominant

Unqualified “dominant-strategy profile” defaults to **weak**.

(You’ll also see “dominant-strategy equilibrium” in the literature.)

Your Turn

Only Player 1's payoffs are shown.

		Player 2		
		E	F	G
Player 1	A	3, ·	2, ·	1, ·
	B	2, ·	1, ·	0, ·
	C	3, ·	2, ·	1, ·
	D	2, ·	0, ·	0, ·

Classify each pair. Then: any weakly dominant strategies?

A vs. B:

A vs. D:

A vs. C:

B vs. D:

So What About Sarah?

Selfish and greedy version:

		Steven	
		Split	Steal
Sarah	Split	50, 50	0, 100
	Steal	100, 0	0, 0

- Steal is a _____ dominant strategy for each player
- So (Steal, Steal) is a _____ dominant-strategy profile

What if both are fair-minded (+75 for equal earnings)?

		Steven	
		Split	Steal
Sarah	Split	125, 125	0, 100
	Steal	100, 0	75, 75

Our Own Example: The Price War

Dos Hermanos and Pitabilities park on the same block every day. Each morning, each owner independently sets a **Normal** price or runs a **Discount**.

- If neither discounts: they split the lunch crowd at healthy margins
- If exactly one discounts: that truck pulls most of the crowd
- If both discount: they split the same crowd at thin margins

Each owner ranks the four outcomes:

(I discount, they don't) $\succ$ (neither) $\succ$ (both) $\succ$ (they discount, I don't)

The Price War

		Pitabilities	
		Normal	Discount
Dos Hermanos	Normal	2, 2	0, <u>3</u>
	Discount	<u>3</u> , 0	<u>1</u> , <u>1</u>

- Discount _____ dominates Normal, for both
- So (Discount, Discount) is a _____ profile

But look at (Normal, Normal):

Definition: Pareto Superiority

Let o and o' be two outcomes.

- o is **strictly Pareto superior** to o' if every player prefers o to o' : $o \succ_i o'$ for all i
- o is **weakly Pareto superior** to o' if every player finds o at least as good, and at least one strictly prefers it

In reduced games, same thing with payoffs: $\pi_i(s) > \pi_i(s')$ for all i , etc.

In the Price War, (Normal, Normal) is strictly Pareto superior to (Discount, Discount).

Yet rational play lands on (Discount, Discount).

The Prisoner's Dilemma

That structure has a name. Bonanno's version:

Doug and Ed are the only two candidates for a best-worker prize, currently tied. Unless one outperforms the other, nobody gets it. Each chooses Normal or Extra effort.

- o_1 : nobody gets the prize, nobody sacrifices family time
- o_2 : Ed gets the prize and sacrifices family time, Doug does not
- o_3 : Doug gets the prize and sacrifices family time, Ed does not
- o_4 : nobody gets the prize, both sacrifice family time

Both are willing to sacrifice family time for the prize, otherwise value it, and are envious:

$o_3 \succ_{\text{Doug}} o_1 \succ_{\text{Doug}} o_4 \succ_{\text{Doug}} o_2$.

The Prisoner's Dilemma

		Ed	
		Normal	Extra
Doug	Normal	2, 2	0, <u>3</u>
	Extra	<u>3</u> , 0	<u>1</u> , <u>1</u>

Extra effort is strictly dominant for both.
(Extra, Extra) is a strict dominant-strategy profile.

Same game as the price war: different story, identical structure.
That is what game theory is for.

Individual vs. Collective Rationality

A strictly dominant strategy is not a suggestion. Playing anything else means a lower payoff *no matter what the other player does*.

So rational individual play $\Rightarrow$ (Extra, Extra), even though both prefer (Normal, Normal).

Couldn't they just agree?

- Non-cooperative game theory assumes agreements are **not binding**
- If I expect you to keep it, I gain by breaking it
- If I expect you to break it, I gain by breaking it too

Let's Work One Out

Three Booths at the North Market

Ann, Ben, and Cal each run a booth. Every Saturday each one independently picks a special: the **Classic** or something **New**.

Three player game: third player picks the **matrix**.

		Ben	
		Classic	New
Ann	Classic	3, 3, 2	4, 3, 1
	New	1, 3, 4	2, 1, 3
		Cal: Classic	

		Ben	
		Classic	New
Ann	Classic	3, 2, 3	4, 0, 2
	New	1, 2, 3	2, 0, 4
		Cal: New	

Ann picks the row, Ben the column, Cal the table. Payoffs in order: Ann, Ben, Cal.

Three Booths at the North Market

1. Does Ann have a dominant strategy? Strict or weak?
2. Does Ben?
3. Does Cal?
4. Is there a dominant-strategy profile?
5. If Ann and Ben play their dominant strategies, what should Cal do?

What Have We Learned, Charlie Brown?

- A **game-frame** is players, strategies, outcomes, and f
- Preferences are a complete, transitive relation $\succsim_i$ on outcomes
- An **ordinal utility function** represents $\succsim_i$; the numbers are arbitrary
- Frame + preferences = **game**; composing gives payoffs $\pi_i = U_i \circ f$
- **Dominance** compares two strategies of one player across all contingencies
- A dominant strategy is best against everything

Next: what if nobody has a dominant strategy?

Reading & Practice

- Bonanno, *Game Theory*, §2.1–2.2
- Exercises: §2.9.1 (frames and preferences), §2.9.2 (dominance)
- Full solutions are in §2.10. Try them before you look